Symbolic Computation to Study Explicit Gröbner Bases and Lattice Path Enumeration

AJ Bu

Rutgers University

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Experimental Mathematics

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The two methods in my dissertation:

1. Writing a code that allows the computer to automatically generate the desired mathematical object
2. Using such codes to produce sufficiently many examples, studying these examples, and identify (then proving) patterns among them

Projects in Dissertation

My dissertation contains the following projects:

1. Finding the Gröbner bases of ideals generated by elementary symmetric polynomials
2. Enumerating infinite families of restricted Dyck paths
3. Enumerating restricted Motzkin paths
4. Enumerating generalized Dyck paths
5. Enumerating generalized Dyck paths with respect to area

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A polynomial f lies in the ideal $I \subset k[x_1, ..., x_n]$ with Gröbner basis G if and only if the remainder on division of f by G is zero.

A **Motzkin path** of length n is a path in the xy-plane from the origin to $(n, 0)$ with steps in $\{(1,1),(1,0),(1,-1)\}$ that never goes below the x-axis.

Introduction to Dyck and Motzkin Paths

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A **Dyck path** is a Motzkin path that avoids flat steps.

Example

The following is a Motzkin path of length 10

$$UUFDUUDFDD$$

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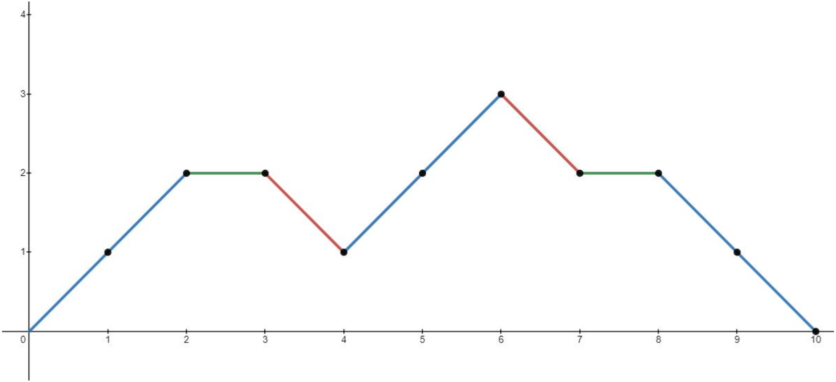

Natural Question: How do we enumerate Motzkin paths?

Use **weight enumerator**:

$$P(t) = \sum_{P \in \mathcal{M}} t^{Length(P)}$$

$$P(t) = 1 + tP(t) + t^2 \left[P(t) \right]^2.$$

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Therefore, the enumerator of each of these gives us

$$P = 1 + tP + t^2 P^2.$$

Restricted Paths

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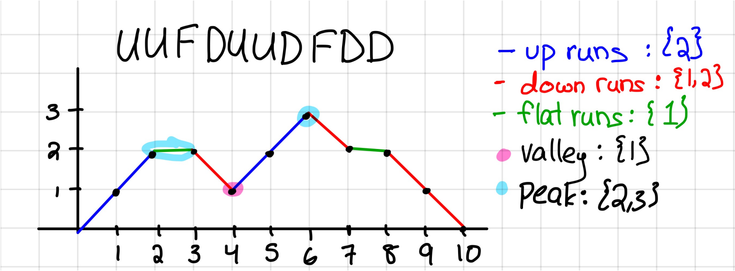

Results from Zeilberger

Created Maple package to find $F(t, X)$ s.t. $F(t, P) = 0$, where P is the weight enumerator of the set of Dyck paths avoiding

- peak heights in A
- valley heights in B
- upward runs in C
- downward runs in D,

where A, B, C, and D are given finite sets containing integers and/or arithmetic progressions.

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1. Can we generalize these findings to Motzkin paths?
2. Can we find identities on the weight enumerator functions for an infinite family of such $\{A, B, C, D\}$?
   - Eg: $C = \{ar + b : r \in \mathbb{N}\}$ for arbitrary $a, b \in \mathbb{N}$

Generalized Zeilberger's procedures to Motzkin paths:

Find $F(t, P)$ directly by generating a finite system of algebraic equation using symbolic dynamic programming and solving it for $F(t, P)$.

- Get new equations and variables by recursively breaking paths down into a concatenation of steps and sub-paths with simpler restrictions
- Use Gröbner bases to get $F(t, P)$

Example Results

The generating function $P(t)$ of the sequence of Motzkin paths with the following restrictions satisfies the given algebraic equation:

Avoiding up-runs of length 1, 2, or 3

$$1 - (t^2 - t + 1)P - t^2(t-1)P^2 + t^8P^4 + t^9P^5 = 0$$

Avoiding odd peak heights and valley heights

$$(t-1)^2 + (t-1)^3P + t^4P^2 = 0$$

Note: Can simultaneously restrict multiple characteristics, where each forbids values given in finite sets containing integers and/or arithmetic progressions

e.g. Avoiding Peak Heights in A and Valley Heights in B

Let $A, B \subset \mathbb{N}$ be finite. Consider the set $\mathcal{P}_{A,B}$ of Motzkin paths avoiding

- peak-heights in A, and
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Consider the following three cases:

1. $0 \in A$
2. $0 \notin A$ and $0 \in B$
3. $0 \notin A$ and $0 \notin B$

Peak at Height 0

Note: The set $\mathcal{F}$ of walks containing only flat-steps contains

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Say a path has a peak at height 0 if and only if it is a flat run.

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Easy to extend to arithmetic progressions $ar + b$ where $a, b \in \mathbb{N}$.

Just use same method on the constant term of the arithmetic progression.

Also wrote programs for avoiding up-runs, down-runs, and flat-runs. Due to time constraints, I will not discuss them here.

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Given a family of Dyck paths $\mathcal{P}$, how do we find an equation satisfied by the weight enumerator $P(t)$?

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2. Translate grammar into equation by replacing every set of $2k$ steps with t^k, and every instance of $\mathcal{P}$ with $P(t)$

Grammatical Family Example

Theorem 1

Let $b < a$ be non-negative integers, and let $\mathcal{P}$ be the set of Dyck paths whose up-run lengths avoid $\{ar + b \mid r \geq 0\}$. Then

$$P(t) = \sum_{\substack{0 \leq k \leq a-1 \\ k \neq b}} t^k \left[P(t)\right]^k + t^a P^{a+1}(t),$$

where $P(t)$ is the weight-enumerator of $\mathcal{P}$.

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Avoid upwards run of length $3r+1$:

$$P = P^4 t^3 + P^2 t^2 + 1$$
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Add together subfamilies:

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$$\implies \qquad P = \sum_{k=0,k\neq b}^{a-1} t^k P^k + t^a P^{a+1}$$

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Use symbolic programming to generate $F(t,X)$ s.t. $F(t,P)=0$, where $P(t)$ is the weight-enumerator for the generalized Dyck paths with steps in a given set S.

E.g. Generalized Dyck paths with steps in $S=\{1,2,-1,-2\}$

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Using our Maple procedure,

```
EqGFt({1,2,-1,-2},P,t)
```

outputs

$$1+(-2t-1)P+t(3t+2)P^2-t^2(2t+1)P^3+P^4t^4.$$

Notation

First let's introduce the following notation:

$$\mathcal{P}_{a,b} = \text{the set of generalized Dyck paths with a set of steps given by } S \text{ that start at } (0, a) \text{ and end at height } b,$$

$$P[a,b](t) = \text{the desired weight-enumerator for the paths in } \mathcal{P}_{a,b}.$$

$$\mathcal{Q}_{a,b} = \text{the subset of } \mathcal{P}_{a,b} \text{ that contains all non-empty paths that stay strictly above the } x-\text{axis, except at an endpoint if } a = 0 \text{ or } b = 0,$$

$$Q[a,b](t) = \text{the desired weight-enumerator for the paths in } \mathcal{Q}_{a,b}.$$

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- We can then use Gröbner bases to get $P(t)$

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To keep track of area as well as the number of paths, we use the following bi-variate weight enumerator:

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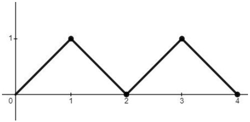

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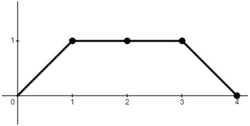

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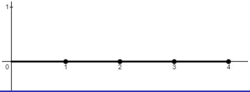

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$$\implies M(t,q) = 1 + tM(t,q) + t^2 qM(t,q)M(qt,q).$$

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Say we know bi-variate polynomials $f(t,q)$, $g(t,q)$, and $h(t,q)$ s.t.

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Brief Description of Process:

1. Plug in $q = 1$
2. Solve for $P(t,1)$
3. Using Taylor series about $q = 1$ and comparing the coefficients of $(q-1)^k$, we can solve for $P^{(k)}(t,1)$

Demonstrate this Process with the Motzkin Paths

$$M(t,q) = 1 + t\,M(t,q) + t^2\,q\,M(qt,q)\,M(t,q).$$

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3. $M(t, 1)$ is the enumerator for Motzkin paths of weight n and has a Taylor series expansion about $t = 1$. Thus
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Area Under Motzkin Paths: Finding $M_q(t,1)$

$$\sum_{k=0}^{n} \frac{(q-1)^k}{k!} M^{(k)}(t,1)$$

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The coefficient of $(q-1)$ on both sides gives:

$$M_q(t,1) = t\, M_q(t,1) + t^2\, M(t,1)\left(t\, M_t(t,1) + 2M_q(t,1) + M(t,1) \right).$$

$$M_q(t,1) = \frac{t^3 M(t,1) M_t(t,1) + t^2 M^2(t,1)}{1 - t - 2t^2 M(t,1)}.$$

We know $M(t,1)$ and can solve for $M_t(t,1)$ by taking the derivative.

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To find $M^{(n)}(t,1)$, we can repeat this process with the coefficient of $M^{(k)}(t,1)$ for $k \leq n$.

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From these, we can also get the average area and the variances.

Take Away

Experimental mathematics is an incredible tool for mathematical research and more people should use it!

Thank You!

Thank you all for attending my defense and for your support!

Thank you to my advisor, Dr. Z, for all of your help and advice throughout my time at Rutgers

Thank you to Angela Gibney for helping me learn about Gröbner bases and for your support

Thank you to my other Committee Members: Prof. Retakh and Bhargav