Explicit Generating Functions for the Sum of the Areas Under Dyck and Motzkin Paths (and for Their Powers)

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Abstract

In this paper, we describe how to find the generating function for the sum of the areas under Motzkin paths of length n through a method that uses dynamical programming, which we show can be expanded for paths with any given set of steps that start and end at height zero and never have a negative height. We then explore the case where, instead of a set of steps, we are given the quadratic functional equation $f(x,q) = P(x,q) + Q(x,q)f(x,q) + R(x,q)f(x,q)f(qx,x)$. We present a fully automated method for finding perturbation expansions of the solutions $f(x,q)$ to such quadratic functional equations and demonstrate this method using Motzkin paths. More importantly, we combine computer algebra with calculus to automatically find $\frac{d^k}{dq^k}\left[f(x,q)\right]\big|_{q=1}$, explicitly expressed in terms of radicals. We use Dyck and Motzkin paths to exemplify how this can be used to find explicit generating functions for the sum of the areas under such paths and for the sum of a given power of the areas.

1 Introduction

In this paper, we are interested in the generating functions for the sum of the areas under paths in the $xy-$plane, with a focus on Dyck and Motzkin paths.

Definition 1. A Motzkin path of length n is a walk in the $xy-$plane from the origin $(0,0)$ to $(n,0)$ with atomic steps $U := (1,1)$, $D := (1,-1)$, and $F := (1,0)$ that never goes below the $x-$axis.

For example, the following are some Motzkin paths of length 4.

$$UDUD \qquad UFFD \qquad UFDF \qquad FUDF \qquad FFFF.$$

The areas of these paths are 2, 4, 3, 1, and 0, respectively.

The bivariate weight enumerator for Motzkin paths with length n and area m satisfies the following functional equation

$$M(x,q) = 1 + xM(x,q) + x^2qM(qx,q)M(x,q).$$

To prove this, let $\mathcal{M}$ denote the set of all Motzkin paths. Note that any path in $\mathcal{M}$ must fall into exactly one of the following cases – the empty path, Motzkin paths that start with a flat step, or Motzkin paths that start with an up step.

If $M \in \mathcal{M}$ is the empty path, then it clearly has both area and length 0. Thus the bivariate weight enumerator is

$$m_0(q,x) = 1$$

If M begins with a flat step F, then we can write

$$M = FM_0,$$

where M_0 must also be a Motzkin path with the same area as M, since it still starts at height 0. Thus, the bivariate weight enumerator for this case of Motzkin paths is

$$m_1(q,x) = xM(q,x)$$

If M begins with the step U, then let D denote the first time M returns to the $x-$axis and write

$$M = UM_1DM_0.$$

M_1 must be a Motzkin path shifted to height 1, and M_0 is a Motzkin path starting at height 0. Since M_0 begins at height 0, the area under the Motzkin path M_0 is the same as the area under the portion of M it represents. Since M_1 is shifted to height 1, however, every step in M_1 has one more unit block below it. Thus, every step x in M_1 must be replaced with qx to get the correct area for that portion of M. Since the extra U and D steps give a combined area of 1, the bivariate weight enumerator for Motzkin paths beginning with an up step is

$$m_2(q,x) = qx^2M(qx,q)M(q,x),$$

resulting in the desired weight enumerator for all Motzkin paths.

Definition 2. A Dyck path of length n is a walk in the $xy-$plane from the origin $(0,0)$ to $(n,0)$ with atomic steps $U := (1,1)$ and $D := (1,-1)$ that never goes below the $x-$axis.

Similarly, the bivariate weight enumerator for Dyck paths with length n and area m satisfies the following functional equation

$$D(x,q) = 1 + x^2qD(qx,q)D(x,q).$$

1.1 Maple Package

This article is accompanied by the Maple package `qEW.txt` and some sample outputs. The accompanying files can be found at

https://ajbu1.github.io/Papers/MotzArea/MotzArea.html.

1.2 Finding the weight enumerator for the area of walks of length n

The procedure `qnwdK(S,K,q)` uses dynamical programming to find the enumerating function for the area of walks with steps in `S` of length $n = 0, \ldots, K$ that start and end at height 0 and never have negative height. This procedure can be used to check the findings presented later in this paper.

First, consider $\mathcal{P}_{m,n}$, the set of paths of length $n \geq 0$ with steps in S that end at height $m \geq 0$ and never have negative height. Let $A_{m,n}(q)$ be the enumerating function for the area of paths in $\mathcal{P}_{m,n}$.

Clearly, for $n = 0$, the empty path gives an area of 0. Thus, the enumerating function is

$$A_{m,0}(q) = 1.$$

For $n = 1$, the only path that can end at height m is the single step $\{(1,m)\}$, which has area $\frac{m}{2}$. Thus,

$$A_{m,1}(q) = \begin{cases} q^{\frac{m}{2}}, & (1,m) \in S \\ 0, & (1,m) \notin S. \end{cases}$$

For $n > 1$, consider each possible final step for any path in $\mathcal{P}_{m,n}$. A step $s \in S$ can be the last step if $m - s \geq 0$ and there exists a path P of length $n-1$ with steps in S that ends at height $m-s$ and never has a negative height. In other words,

$$\mathcal{P}_{m,n} = \{Ps | s \in S,\ m - s \geq 0,\ P \in \mathcal{P}_{m-s,n-1}\}.$$

The area under the last step $(1,s)$ is $\frac{2m-s}{2}$. Thus, the weight enumerator for the area of paths in $\mathcal{P}_{m,n}$ is

$$A_{m,n}(q) = \sum_{\substack{s\in S\\ m-s\geq 0}} q^{\frac{2m-s}{2}} A_{m-s,n-1}(q).$$

This process is implemented in the procedure `qnmwd(S,n,m,q)`, which is then used in `qnwdK(S,K,q)`. For example, looking at Motzkin paths,

```
qnwdK({[1,1],[1,0],[1,-1]},5,q)
```

outputs

$$[1, 1, q+1, q^2+2q+1, q^4+q^3+3q^2+3q+1, q^6+2q^5+3q^4+4q^3+6q^2+4q+1] \quad .$$

Note that, to avoid negative height, any Motzkin path must end with $D = (1,-1)$ or $F = (1,0)$. Since the paths end at height 0, the area under these steps are $\frac{1}{2}$ and 0, respectively. Thus,

$$A_{0,n} = q^{\frac{1}{2}} A_{1,n-1} + A_{0,n-1}.$$

Breaking down the algorithm described above to find the first four terms of this outputted list,

- The only path of length 1 is $[F] = [(1,0)]$, so

$$A_{0,1}(q) = 1.$$

- Since $\mathcal{P}_{1,1} = \{[U]\} = \{[(1,0)]\}$, it follows that $A_{1,1}(q) = q^{\frac{1}{2}}$. Thus,

$$\begin{aligned} A_{0,2} &= q^{\frac{1}{2}} A_{1,1}(q) + A_{0,1}(q) \\ &= q+1. \end{aligned}$$

- For paths of length 3 ending with $D = (1,-1)$, note that

$$\mathcal{P}_{1,2} = \{FU, UF\} = \{[(1,0),(1,1)],[(1,1),(1,0)]\},$$

  and so $A_{1,2} = q^{\frac{1}{2}} + q^{\frac{3}{2}}$. Thus,

$$\begin{aligned} A_{0,3} &= q^{\frac{1}{2}} A_{1,2}(q) + A_{0,2}(q) \\ &= q^{\frac{1}{2}}(q^{\frac{1}{2}} + q^{\frac{3}{2}}) + q + 1 \\ &= q^2 + 2q + 1. \end{aligned}$$

- For paths of length 4, note that

$$\begin{aligned} \mathcal{P}_{1,3} &= \{FFU, FUF, UFF, UDU\} \\ &= \{[(1,0),(1,0),(1,1)],[(1,1),(1,-1),(1,1)],[(1,0),(1,1),(1,0)],[(1,1),(1,0),(1,0)]\}. \end{aligned}$$

  Therefore, $A_{1,3} = q^{\frac{1}{2}} + 2q^{\frac{3}{2}} + q^{\frac{5}{2}}$, and

$$\begin{aligned} A_{0,4} &= q^{\frac{1}{2}} A_{1,3}(q) + A_{0,3}(q) \\ &= q^4 + 3q^2 + 3q + 1. \end{aligned}$$

2 Perturbation Expansions of Solutions to Quadratic Functional Equations

Suppose that a function $f(x,q)$ satisfies the functional equation

$$f(x,q) = P(x,q) + Q(x,q)f(x,q) + R(x,q)f(x,q)f(qx,q),$$

for given bivariate polynomials $P(x,q)$, $Q(x,q)$, and $R(x,q)$. To find $f(x,q)$ up to degree k in x, first set $f_0(x,q) := 1$. For $i > 0$, let

$$f_i(x,q) = P(x,q) + Q(x,q)f_{i-1}(x,q) + R(x,q)f_{i-1}(qx,q)f_{i-1}(x,q),$$

and find $n > 0$ such that $f_n(x,q)$ and $f_{n+1}(x,q)$ agree up to degree k in x. Note that for any $i > n$, $f_n(x,q)$ and $f_i(x,q)$ will also agree up to degree k in x, and thus $f(x,q)$ and $f_n(x,q)$ will as well.

This process is implemented in the Maple procedure `SeqF1(P,Q,R,q,x,K)`, which inputs bivariate polynomials P, Q, and R, variables q and x, and a non-negative integer K, and outputs $f(x,q)$ up to degree K in x. For example, the weight enumerator for the area under Dyck paths of lengths $k = 0,\dots,8$ is found by

`SeqF1(1,0,x^2*q,q,x,8)`,

which outputs

$$1+x^2q+(q^4+q^2)x^4+(q^9+q^7+2q^5+q^3)x^6+(q^{16}+q^{14}+2q^{12}+3q^{10}+3q^8+3q^6+q^4)x^8.$$

The weight enumerator for the area under Motzkin paths of lengths $k = 0,\dots,5$ is found by

`SeqF1(1,x,x^2*q,q,x,5)`,

which outputs

$$1+x+(q+1)x^2+(q^2+2q+1)x^3+(q^4+q^3+3q^2+3q+1)x^4+(q^6+2q^5+3q^4+4q^3+6q^2+4q+1)x^5.$$

Note that the coefficient of x^k and the k-th term of the list output by `qnwdK([1,1],[1,0],[1,-1]),5,q)` are equal, as desired. Using this method, the expression is found through the following calculations

$$\begin{aligned}
f_0(x,q) &= 1\\
f_1(x,q) &= 1 + xf_0(x,q) + x^2qf_0(x,q)f_0(qx,q)\\
&= 1 + x + x^2q\\
f_2(x,q) &= 1 + xf_1(x,q) + x^2qf_1(x,q)f_1(qx,q)\\
&= 1 + x + (q+1)x^2 + (q^2+2q)x^3 + (q^4+2q^2)x^4 + (q^4+q^3)x^5 + \dots\\
f_3(x,q) &= 1 + xf_2(x,q) + x^2qf_2(x,q)f_2(qx,q)\\
&= 1 + x + (q+1)x^2 + (q^2+2q+1)x^3 + (q^4+q^3+3q^2+3q)x^4\\
&\quad + (q^6+2q^5+2q^4+3q^3+5q^2)x^5 + \dots\\
f_4(x,q) &= 1 + xf_3(x,q) + x^2qf_3(x,q)f_3(qx,q)\\
&= 1 + x + (q+1)x^2 + (q^2+2q+1)x^3 + (q^4+q^3+3q^2+3q+1)x^4\\
&\quad + (q^6+2q^5+3q^4+4q^3+6q^2+4q)x^5 + \dots
\end{aligned}$$

$$\begin{aligned}
f_5(x,q) &= 1 + xf_4(x,q) + x^2 q f_4(x,q) f_4(qx,q)\\
&= 1 + x + (q+1)x^2 + (q^2+2q+1)x^3 + (q^4+q^3+3q^2+3q+1)x^4\\
&\quad + (q^6+2q^5+3q^4+4q^3+6q^2+4q+1)x^5 + \dots\\
f_6(x,q) &= 1 + xf_4(x,q) + x^2 q f_4(x,q) f_4(qx,q)\\
&= 1 + x + (q+1)x^2 + (q^2+2q+1)x^3 + (q^4+q^3+3q^2+3q+1)x^4\\
&\quad + (q^6+2q^5+3q^4+4q^3+6q^2+4q+1)x^5 + \dots
\end{aligned}$$

Since $f_5(x,q)$ and $f_6(x,q)$ agree up to degree 5 in x, the procedure outputs

$$1+x+(q+1)x^2+(q^2+2q+1)x^3+(q^4+q^3+3q^2+3q+1)x^4+(q^6+2q^5+3q^4+4q^3+6q^2+4q+1)x^5.$$

3 Finding $\frac{d^k}{dq^k}\left[f(x,q)\right]\Big|_{q=1}$

If the weight enumerator of a set of paths is satisfied by the following functional equation

$$f(x,q) = P(x,q) + Q(x,q)f(x,q) + R(x,q)f(x,q)f(qx,q)$$

for some given bivariate polynomials $P(x,q)$, $Q(x,q)$, and $R(x,q)$, then plugging in $q=1$ gives

$$f(x,1) = P(x,1) + Q(x,1)f(x,1) + R(x,1)f(x,1)^2,$$

which we can use to solve for $f(x,1)$. The order n Taylor polynomial of $f(x,q)$ about $q=1$ satisfies

$$\sum_{k=0}^{n}\frac{(q-1)^k}{k!}f^{(k)}(x,1) = P+Q\sum_{k=0}^{n}\frac{(q-1)^k}{k!}f^{(k)}(x,1)+R\sum_{k=0}^{n}\frac{(q-1)^k}{k!}f^{(k)}(x,1)\sum_{k=0}^{n}\frac{(q-1)^k}{k!}f^{(k)}(qx,1),$$

where $f^{(k)}(x,q) = \frac{d^k}{dq^k}f(x,q)$. Looking at the coefficient of $(q-1)^k$, we can express $f^{(k)}(x,1)$ as the sum of derivatives $f^{(\ell)}(x,1)$ where $\ell < k$ and derivatives of functions of x with respect to x. Since we have an expression for $f(x,1)$, we can simply compute any order derivative with respect to x as well as $f_q(x,1)$. Thus, to find $f^{(n)}(x,1)$, we can repeat this process with the coefficient of $f^{(k)}(x,1)$ for $k = 1,\dots,n$.

This process is implemented by the procedure `DerK(P,Q,R,q,x,K,f)`, which outputs a list whose k-th entry is $\frac{d^{k-1}}{dq^{k-1}}\left[f(x,q)\right]\Big|_{q=1}$. Rather than outputting algebraic equations as seen in [2], this procedure produces closed-form expressions in terms of radicals.

3.1 Motzkin Paths

As previously noted, the Motzkin paths satisfy the following functional equation

$$M(x,q) = 1 + xM(x,q) + x^2 qM(qx,q)M(x,q).$$

Solving this functional equation for $q=1$, we get that

$$M(x,1) = \frac{1-x+\sqrt{-3x^2-2x+1}}{2x^2} \qquad \text{or} \qquad M(x,1) = \frac{1-x-\sqrt{-3x^2-2x+1}}{2x^2}.$$

Since only the second equation has a Taylor series expansion about $x=0$, we know that this is $M(x,1)$. Now, for finding the first derivative, note that

$$\sum_{k=0}^{n}\frac{(q-1)^k}{k!}M^{(k)}(x,1) = 1+x\sum_{k=0}^{n}\frac{(q-1)^k}{k!}M^{(k)}(x,1)+qx^2\sum_{k=0}^{n}\frac{(q-1)^k}{k!}M^{(k)}(x,1)\sum_{k=0}^{n}\frac{(q-1)^k}{k!}M^{(k)}(qx,1)$$

The coefficient of $q-1$ on both sides give us

$$M_q(x,1) = xM_q(x,1) + x^2M(x,1)\Big(xM_x(x,1) + 2M_q(x,1) + M(x,1)\Big).$$

Therefore,

$$M_q(x,1) = \frac{x^3M(x,1)M_x(x,1) + x^2M^2(x,1)}{1-x-2x^2M(x,1)}.$$

Plugging in $M(x,1) = \frac{1-x-\sqrt{-3x^2-2x+1}}{2x^2}$, we get

$$M_q(x,1) = \frac{\left(x-1+\sqrt{-3x^2-2x+1}\right)^2}{4x^2(-3x^2-2x-1)}$$

To find $M^{(n)}(x,1)$, we can repeat this process with the coefficient of $M^{(k)}(x,1)$ for $k \le n$.

In a little over 2 seconds,

DerK(1,x,x^2*q,q,x,10,f),

can output the list whose entries are $M^{(k)}(q,1) := \frac{d^k}{dq^k}\left[M(x,q)\right]|_{q=1}$ for $k = 0,\ldots,10$. For example, looking at the first two terms of the output, we have

$$M(x,1) = \frac{1-x-\sqrt{-3x^2-2x+1}}{2x^2}, \qquad \text{and}$$
$$M_q(x,1) = \frac{\left(1-x-\sqrt{-3x^2-2x+1}\right)^2}{4x^2(-3x^2-2x+1)}$$

The Maclaurin Series of $M(x,1)$ is

$$1+x+2x^2+4x^3+9x^4+21x^5+51x^6+127x^7+323x^8+835x^9+2188x^{10}+5798x^{11}+15511x^{12}+O(x^{13}),$$

and it is the weight enumerator of the number of Motzkin paths of length n, which is A001006 on [5], https://oeis.org/A001006. The Maclaurin series of $M_q(x,1)$ is

$$x^2+4x^3+16x^4+56x^5+190x^6+624x^7+2014x^8+6412x^9+20219x^{10}+63284x^{11}+196938x^{12}+O(x^{13}),$$

which is the weight enumerator of the total area under all Motzkin paths of length n and A057585 on [5], https://oeis.org/A057585.

We also get higher factorial moments. For example,
$M_{qq}(x,1) = 1/2(6(-3x^2-2x+1)^{1/2}x^2 + 9x^2 - (-3x^2-2x+1)^{1/2}x + 6x + 3(-3x^2-2x+1)^{1/2} - 3)(-1+x+(-3x^2-2x+1)^{1/2})/(3x^2+2x-1)^3$,

and

$M_{qqq}(x,1) = -3/2(9(-3x^2-2x+1)^{1/2}x^4 - 9x^5 + 18(-3x^2-2x+1)^{1/2}x^3 + 51x^4 - 23(-3x^2-2x+1)^{1/2}x^2 - 19x^3 + 4(-3x^2-2x+1)^{1/2}x + 29x^2 - 4(-3x^2-2x+1)^{1/2} - 8x + 4)(x-1+(-3x^2-2x+1)^{1/2})/(3x^2+2x-1)^4$.

The Maclaurin series of $M_{qq}(x,1)$ is

$$2x^3+24x^4+142x^5+720x^6+3224x^7+13478x^8+53508x^9+204698x^{10}+O(x^{11}),$$

and the Maclaurin series of $M_{qqq}(x,1)$ is

$$30x^4+336x^5+2742x^6+17268x^7+95388x^8+477900x^9+2235876x^{10}+O(x^{11}).$$

The weight enumerator for the sum of the squares of the areas of Motzkin paths of length n is given by the Maclaurin series of $M_{qq}(x,1)+M_q(x,1)$,

$$x^2+6x^3+40x^4+198x^5+910x^6+3848x^7+15492x^8+59920x^9+224917x^{10}+O(x^{11}),$$

and the weight enumerator for the sum of the cubes of the areas of Motzkin paths of length n is given by the Maclaurin series of $M_{qqq}(x,1)+3M_{qq}(x,1)+M_q(x,1)$,

$$x^2+10x^3+118x^4+818x^5+5092x^6+27564x^7+137836x^8+644836x^9+2870189x^{10}+O(x^{11}).$$

None of these appear on OEIS as of September 12, 2023.

3.2 Dyck Paths

Looking at Dyck paths, we input

```
DerK(1,0,x^2*q,q,x,10,f).
```

The first four terms of the output gives

$$D(x,1)=\frac{1-\sqrt{1-4x^2}}{2x^2}$$

$$D_q(x,1)=\frac{(1-\sqrt{1-4x^2})^2}{16x^4-4x^2}$$

$$D_{qq}(x,1)=\frac{(8x^2\sqrt{1-4x^2}+12x^2+3\sqrt{1-4x^2}-3)(-1+\sqrt{1-4x^2})}{2}$$

$$D_{qqq}(x,1)=\frac{-6(4x^4\sqrt{-4x^2+1}+16x^4-7x^2\sqrt{-4x^2+1}+7x^2-\sqrt{-4x^2+1}+1)(-1+\sqrt{-4x^2+1})}{(4x^2-1)^4}$$

The Maclaurin series of $D(x,1)$ is

$$1+x^2+2x^4+5x^6+14x^8+42x^{10}+132x^{12}+429x^{14}+1430x^{16}+O(x^{18}),$$

which is the weight enumerator of all Dyck paths of length n and A000108 on [5], https://oeis.org/A000108. The Maclaurin series of $D_q(x,1)$ is

$$x^2+6x^4+29x^6+130x^8+562x^{10}+2380x^{12}+9949x^{14}+41226x^{16}+O(x^{18}),$$

the weight enumerator for the total area of all Dyck paths of length n, which is A008549 on [5], https://oeis.org/A008549.
The Maclaurin series of $D_{qq}(x,1)$ is

$$14x^4+160x^6+1226x^8+7864x^{10}+45564x^{12}+247136x^{14}+1279810x^{16}+6404424x^{18}+O(x^{20}),$$

and the Maclaurin series of $D_{qqq}(x,1)$ is

$$24x^4+840x^6+11736x^8+114744x^{10}+922224x^{12}+6541776x^{14}+42543480x^{16}+259525464x^{18}+O(x^{20}).$$

The weight enumerator for the sum of the squares of the areas of Dyck paths of length n is given by the Maclaurin series of $D_{qq}(x,1)+D_q(x,1)$,

$$x^2+20x^4+189x^6+1356x^8+8426x^{10}+47944x^{12}+257085x^{14}+1321036x^{16}+O(x^{18}).$$

The weight enumerator for the sum of the cubes of the areas of Dyck paths of length n is given by the Maclaurin series of $D_{qqq}(x,1)+3D_{qq}(x,1)+D_q(x,1)$,

$$x^2+72x^4+1349x^6+15544x^8+138898x^{10}+1061296x^{12}+7293133x^{14}+46424136x^{16}+O(x^{18}).$$

None of these appear on OEIS as of September 12, 2023.

4 Conclusion

In this paper, we demonstrate how to use dynamical programming to find the weight enumerator for the area paths of length n with steps in a given set S that start and end at height 0 and never have negative height. We also describe how to find the weight enumerator for such paths when, instead of a set of steps S, we are given bivariate polynomials $P(x,q)$, $Q(x,q)$, and $R(x,q)$ such that the weight enumerator $f(x,q)$ satisfies

$$f(x,q) = P(x,q) + Q(x,q)f(x,q) + R(x,q)f(x,q)f(xq,q).$$

We then present a method for finding $f^{(k)}(x,1) := \frac{d^k}{dq^k}[f(x,q)]\big|_{q=1}$.

These methods are fully automated in the accompanying Maple package `qEW.txt`, displaying how the power of computer algebra and using calculus allows us to generate quite a few moments. In the paper, we demonstrate these methods with the bivariate weight enumerators for both Motzkin paths and Dyck paths with length n and area m. Moreover, we show how these procedures can be used to produce the Maclaurin series of $\frac{d^k}{dq^k}[f(x,q)]\big|_{q=1}$, allowing us to find the generating function for the total area under such paths of length n as well as for the sum of a given power of the areas.

For further study, we can look at the average areas and the variance. Given a family of paths, let $a_0(n)$ be the number of such paths of length n, $a_1(n)$ be the total area under such paths of length n, and $a_2(n)$ be the sum of the squares of the areas under such paths of length n. Using the accompanying Maple package `qEW.txt`, we can generate 10,000 (or more) terms of the sequences of the average areas $\left\{\frac{a_1(n)}{a_0(n)}\right\}$ and the variances $\left\{\frac{a_2(n)}{a_0(n)} - \left(\frac{a_1(n)}{a_0(n)}\right)^2\right\}$ and use numerics for the asymptotics.

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