Rutgers University: Algebra Written Qualifying Exam

January 2019: Problem 3 Solution

Exercise. Let A and B be operators in complex finite-dimensional vector space such that $AB - BA = B$.

(a) Prove that for all integer $k > 0$ there holds $AB^k - B^kA = kB^k$.

Solution.

Use induction:
Base case: $k = 1$ we are given $AB - BA = B$.
Now suppose $AB^k - B^kA = kB^k$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned} & AB^k = B^kA + kB^k \\ \implies \quad & BAB^k = B\left(B^kA + kB^k\right) \\ & \qquad = B^{k+1}A + kB^{k+1} \\ \implies \quad & (AB - B)B^k = B^{k+1}A + kB^{k+1}, \qquad \text{since } BA = AB - B \\ \implies \quad & AB^{k+1} - B^{k+1} = B^{k+1}A + kB^{k+1} \\ \implies \quad & AB^{k+1} - B^{k+1}A = (k+1)B^{k+1} \end{aligned}$$

So, by induction, for all integers $n > 0$,

$$AB^n - B^nA = nB^n$$

(b) Prove that operator B is nilpotent.

Solution.

Let $k \in \mathbb{N}$ be arbitrary.

$$\begin{aligned} kTr(B^k) &= Tr(kB^k) \\ &= Tr(AB^k - B^kA) \\ &= Tr(AB^k) - Tr(B^kA) \\ &= Tr(AB^k) - Tr(AB^k) \\ &= 0 \\ \implies Tr(B^k) &= 0 \text{ for all integers } k \geq 0 \end{aligned}$$

Thus, B is nilpotent.