Rutgers University: Algebra Written Qualifying Exam

January 2019: Problem 1 Solution

Exercise. Let P be a Sylow $p-$subgroup of a finite group G and H be a normal subgroup in G

(a) Prove that the intersection of P and H is a Sylow $p-$subgroup in H.

Solution.

Let P be a Sylow p-subgroup of G and $H \lhd G$.
$|P| = p^n$ for some prime p such that $p^n \mid |G|$ but $p^{n+1} \nmid |G|$

Case 1: $p \nmid |H|$

If $p \nmid |H|$, then the Sylow p-subgroup of H is $\{e\}$.
$P \cap H = \{e\}$ since the elements of P must have order p^j, $0 \leq j \leq n$, and the order of the elements of H must divide $|H|$
Since $p \nmid |H|$, both of the conditions hold only when $j = 0$.
Thus, $p \cap H = \{e\}$, the Sylow p-subgroup of H.

Case 2: $p \mid |H|$

Then $|H| = p^k m$ for some k s.t. $1 \leq k \leq n$ and $gcd(p, m) = 1$
Since P and H are groups, $P \cap H$ is also a group.
Moreover, $P \cap H \subseteq P$ and $P \cap H \subseteq H$.
Thus, $|P \cap H| \mid |P|$ and $|P \cap H| \mid |H|$
So, $|P \cap H| = p^\ell$ where $0 \leq \ell \leq k$
Sylow II states that any subgroup of order p^i is contained in a Sylow subgroup.
So $\exists$a Sylow p-sbugroup of H, say Q, such that $|P \cap H| \subseteq Q$ and $|Q| = p^k$
Also by Sylow II, any two Sylow p-subgroups of G are conjugate
So, $\exists g \in G$ s.t. $Q \subseteq gPg^{-1}$
(note: this is equivalent to saying Q is contained in a Sylow p-subgroup of G)
$\implies g^{-1}Qg \subseteq P$
Also, $Q \subseteq H$ and since H is a **normal** subgroup of G, $gHg^{-1} = H$
$\implies g^{-1}Qg \subseteq H$.
Therefore, $g^{-1}Qg \subseteq P \cap H$.
$|Q| = |g^{-1}Qg|$ since they are conjugates
$\implies |Q| = |g^{-1}Qg| \leq |P \cap H|$
But $P \cap H \subseteq Q$.
Thus, it follows that $P \cap H = Q$, i.e. $P \cap H$ is a Sylow p-subgroup of H

(b) Find an example showing that for non-normal subgroups H the statement (a) may not be valid.

Solution.

Let $G = S_3$.
$|G| = 6$ so a Sylow 2-subgroup is a subgroup of order 2.
Let $P = \{(\ \ ), (1\ \ 2)\}$, $H = \{(\ \ ), (1\ \ 3)\}$.
Then P is a Sylow 2-subgroup of G and H is a subgroup of G.
But $P \cap H = \{(\ \ )\}$ is **not** a Sylow 2-subgroup of H.