Rutgers University: Real Variables and Elementary Point-Set Topology Qualifying Exam

January 2018: Problem 4 Solution

Exercise. Let λ denote Lebesuge measure on $\mathbb{R}$. Let $x \in \mathbb{R}$, and let $f : x \mapsto f(x) \in \mathbb{R}$ be Lebesgue measurable. For Borel sets $B \subset \mathbb{R}$, define

$$\mu(B) = \lambda(\{x : f(x) \in B\}).$$

Show that μ is a measure, and that

$$\int_{\mathbb{R}} g(y)d\mu(y) = \int_{\mathbb{R}} (g \circ f)(x) d\lambda(x)$$

for all g such that the integrals make sense.

Solution.

μ is a **measure** if

(a) $\mu : \mathcal{M} \to [0, \infty]$

(b) $\mu(\emptyset) = 0$

(c) $\{E_j\}_1^\infty$ disjoint $\implies \mu\left(\bigcup_1^\infty E_j\right) = \sum_1^\infty \mu(E_j)$

(a) $\mu(B) = \lambda(\{x : f(x) \in B\})$ and λ a measure, so $\lambda(\{x : f(x) \in B\}) \geq 0$
$\implies \mu(B) \geq$, i.e. $\mu : B_{\mathbb{R}} \to [0, \infty]$

(b) $\mu(\emptyset) = \lambda(\{x : f(x) \in \emptyset\}) = \lambda(\emptyset) = 0$ since λ a measure

(c) For $\{E_j\}_1^\infty$ disjoint, let $F_j = \{x : f(x) = E_j\}$ so $\mu(E_j) = \lambda(F_j)$.

So,

$$\begin{aligned} x \in F_j &\implies f(x) \in E_j \\ &\implies f(x) \notin E_k && \text{for } k \neq j \text{ since } \{E_j\} \text{ disjoint.} \\ &\implies x \notin f_k && \text{for } k \neq j \\ &\implies \{F_j\}_1^\infty \text{ is disjoint} \end{aligned}$$

$$\begin{aligned} \mu\left(\bigcup_1^\infty E_j\right) &= \lambda\left(\{x : f(x) \in \bigcup_1^\infty E_j æ\right)) \\ &= \lambda\left(\bigcup_1^\infty \{x : f(x) \in E_j\}\right) \\ &= \lambda\left(\bigcup_1^\infty F_j\right) \\ &= \sum_1^\infty \lambda(F_j) \\ &= \sum_1^\infty \mu(E_j) \end{aligned}$$

Thus, μ is a measure

Solution.

Show that

$$\int_{\mathbb{R}} g(y)d\mu(y) = \int_{\mathbb{R}} (g \circ f)(x)d\lambda(x)$$

for all g such that the integrals make sense.
Look at simple functions.
Let $\{\phi_n\}$ be a sequence of simple functions converging pointwise almost everywhere monotonically up to g, so by the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int \phi_n d\lambda = \int g d\lambda$$

$$\phi_n = \sum_{j=1}^{k_n} \alpha_{j,n}\chi(E_{j,n})$$

$$\implies \qquad \int \phi d\lambda = \sum_{j=1}^{k_n} \alpha_{j,n}\lambda(E_{j,n})$$

$$= \sum_{j=1}^{k_n} \alpha_{j,n} \int E_{j,n} f d\mu$$

$$\int f\left(\sum_{j=1_n^k} \alpha_{j,n}\chi(E_{j,n})\right) d\mu = \int f \circ \phi_n d\mu \qquad f \circ \phi_n \text{ converges pointwise to } f \circ g$$

By MCT

$$\int g d\lambda = \lim_{n\to\infty} \int f \circ \phi_n d\mu$$

$$= \int f \circ g d\mu$$