Rutgers University: Complex Variables and Advanced Calculus Written Qualifying Exam

January 2018: Problem 4 Solution

Exercise. Define $D = \{z \in \mathbb{Z} : 2 < |z| < 3\}$. Let f be a holomorphic function over D that is continuous over $\overline{D}$.

(a) Suppose that $\max_{|z|=2} |f(z)| \leq 2$ and $\max_{|z|=3} |f(z)| \leq 3$. Prove that $|f(z)| \leq |z|$ on D

Solution.

Consider the function $\frac{f(z)}{z}$.

We are given that $\left|\frac{f(z)}{z}\right| \leq 1$ on δD.

Also, since $f(z)$ is holomorphic on D and the only singularity of $\frac{f(z)}{z}$ is at $z = 0$,
it follows that $\frac{f(z)}{z}$ is holomorphic on D.

By the **Maximum Modulus Principle**, since $\frac{f(z)}{z}$ is holomorphic on the connected open set $D \subseteq \mathbb{C}$, $\left|\frac{f(z)}{z}\right|$ attains its maximum on δD

Thus, $|\frac{f(z)}{z}| \leq 1$ for all $z \in D$

$\implies |f(z)| \leq |z|$ on D

(b) Suppose $|f(z)| = |z|$ for $|z| = 2$ and $|z| = 3$. Suppose furthermore that $f(z)$ does not have any zeros in D. Prove that $f(z) = e^{i\theta}z$ for some constant $\theta \in [0, 2\pi]$.

Solution.

By the **minimum modulus principle** since $\frac{f(z)}{z}$ is holomorphic in D (a bounded domain), continuous up to the boundary of D, and nonzero at all points, $\left|\frac{f(z)}{z}\right|$ takes its minimum on the boundary of D.

$\implies \left|\frac{f(z)}{z}\right| \geq 1$ on D

But we also know that $|f(z)| \leq |z|$ by part (a)

$\implies |f(z)| = |z|$ on D

$\implies f(z) = zg(z)$ for some $g(z)$ such that $|g(z)| = 1$ on $\overline{D}$.

$g(z) = \frac{f(z)}{z}$ is holomorphic on D and, for any $z_0 \in D$, $|g(z_0)| = 1 \geq |g(z)|$ for all $z \in D$.

(i.e. $|g(z)|$ attains its maximum in the compact nonempty set $\overline{D}$ inside the boundary)

$\implies$ By the maximum modulus principle, g is constant

Thus $g(z) = e^{i\theta}$ for some $\theta \in [0, 2\pi]$

$\implies f(z) = e^{i\theta}z$