Rutgers University: Real Variables and Elementary Point-Set Topology Qualifying Exam

January 2017: Problem 1 Solution

Exercise. The sum $A+B$ of two subsets of $\mathbb{R}^n$ is

$$A+B=\{x+y : x\in A, y\in B\}$$

(a) Show if A is closed and B is compact, then $A+B$ is closed.

Solution.

Let $(a_n)\subseteq A$ and $(b_n)\subseteq B$. Also let $(z_n)=(a_n+b_n)\subseteq A+B$ s.t. $(z_n)\to z$.
We want to show that $z\in A+B$.
Since A is closed,

$$\underline{\textbf{if}}\ (a_n) \text{ converges then } (a_n)\to a\in A$$

Since B is compact,

$$\exists\text{a subsequence } (b_{n_k})\subseteq(b_n) \text{ s.t. } (b_{n_k})\to b\in B$$

$$\begin{aligned} z_n = a_n + b_n \quad &\implies \quad a_n = z_n - b_n \\ &\implies \quad a_{n_k} = z_{n_k} - b_{n_k} \end{aligned}$$

Since (z_n) converges, $(z_{n_k})\subset(z_n)$ converges.

$$\begin{aligned} &\implies \quad (z_{n_k}-b_{n_k}) \text{ converges} \\ &\implies \quad (a_{n_k}) \text{ converges} \end{aligned}$$

Since $(a_{n_k})\subset A$ and A is closed, $(a_{n_k})\to a\in A$.
Thus, $z=a+b\in A+B$, and $A+B$ is closed.

(b) Show sum $A+B$ of two compact subsets of $\mathbb{R}^n$ is compact

Solution.

Let $(z_n)\subset A+B$ be a sequence. Then

$$(z_n)=(a_n+b_n)\subset A+B \qquad \text{and} \qquad (a_n)\subset A \qquad \text{and} \qquad (b_n)\subset B$$

Since A and B are compact, $\exists$subsequences $(a_{n_k})\subset(a_n)$ and $(b_{n_k})\subset(b_n)$ s.t.

$$\begin{aligned} & (a_{n_k})\to a\in A && \text{and} \quad (b_{n_k})\to b\in B \\ \implies\ & (a_{n_k}+b_{n_k})\to a+b\in A+B && \\ \implies\ & (z_{n_k})\to a+b\in A+B && \text{and} \quad (z_{n_k})\subset(z_n) \\ \implies\ & A+B \text{ is compact} && \end{aligned}$$

(c) Show the sum of two closed sets is not necessarily closed.

Solution.

$$\begin{aligned} & A = \mathbb{N} \quad \text{and} \quad B = \{-n + \frac{1}{n} : n \in \mathbb{N}\} \quad \text{are both closed} \\ \implies & \left(\frac{1}{n}\right) \subset A + B \quad \text{and} \quad \left(\frac{1}{n}\right) \to 0 \notin A + B \\ \implies & A + B \text{ is not closed} \end{aligned}$$