Rutgers University: Algebra Written Qualifying Exam

January 2017: Problem 5

Exercise. Let p be prime and G be a $p-$group. Let X be a finite set with $|X|$ not divisible by p. Suppose G acts on X. Prove that $\exists x \in X$ with orbit $G \cdot x = \{x\}$, that is, the action of G on X must have at least one fixed point.

Solution.

$orb(x) = \{gx : g \in G\} \subseteq X$ and $stab(x) = \{g \in G : gx = x\}$

Orbit Stabilizer Theorem: If G is a finite group acting on X then

$$|G| = |orb(x)||stab(x)| \qquad \forall x \in X$$

Assume, **for contradiction,** that $\forall x \in X$, $orb(x) \neq \{x\}$. In other words, $|orb(x)| > 1$. By the Orbit Stabilizer Theorem,

$$\begin{aligned} p^k = |G| = |orb(x)||stab(x)| \quad &\implies \quad |orb(x)| \mid p^k \\ &\implies \quad |orb(x)| = p^j \qquad 0 < j \leq k \qquad \forall x \in X \end{aligned}$$

Class Equation: If G a group acting on finite set X then if

$$X_0 = \{\text{fixed points of the action } G \text{ on } X\}, \qquad \mathcal{O}_1, \dots, \mathcal{O}_r = \text{orbits of size greater than } 1$$

and for each $\mathcal{O}_i$, let $x_i \in \mathcal{O}_i$ and G_i be the stabilizer of x_i in G, i.e. $G_i = \{g \in G : gx_i = x_i\}$, Then

$$|X| = |X_0| + \sum_{i=1}^{r} \frac{|G|}{|G_i|}$$

$|X_0| = 0$ by our assumption, and $\displaystyle\sum_{i=1}^{r} \frac{|G|}{|G_i|} = \sum_{i=1}^{r} |\mathcal{O}_i|$ by the Orbit Stabilizer Theorem

$$\implies |X| = |X_0| + \sum_{i=1}^{r} \frac{|G|}{|G_i|} = \sum_{i=1}^{r} |\mathcal{O}_i|, \quad \text{where } \mathcal{O}_1, \dots, \mathcal{O}_r \text{ are the orbits for } G \text{ acting on } x_i$$

We showed earlier that $|orb(x)| = p^j$ where $0 < j \leq k$ for all $x \in X$

$$\begin{aligned} &\implies |\mathcal{O}_i| = pm_i, \qquad \text{for } i = \{1, \dots, r\} \\ &\implies |X| = \sum_{i=1}^{r} |\mathcal{O}_i| \\ &\qquad = \sum_{i=1}^{r} pm_i \\ &\qquad = p\sum_{i=1}^{r} m_i \\ &\implies p \mid |X|, \text{ which is a contradiction since } |X| \text{ is not divisible by } p \\ &\implies \exists x \in X \text{ with } orb(x) = \{x\} \end{aligned}$$