Rutgers University: Algebra Written Qualifying Exam

January 2017: Problem 2

Exercise. Prove that if the ring of polynomials $R[x]$ over a commutative domain R with identity is a principal ideal ring, then R is a field.

Solution.

R is a **commutative domain:** commutative ring with an identity and no zero divisors

$R[x]$ is a **principal ideal ring:** every ideal is generated by a single element.

And R is a **field** if it is a commutative ring and R^* is a subgroup of $(R, \cdot, 1)$

Want to show: R has inverses under multiplication.

Let $r \in R$ s.t. $r \neq 0$.

$\langle r, x \rangle$ is an ideal in $R[x]$

Since $\langle r, x \rangle$ is an ideal in $R[x]$ and $R[x]$ is a principal ideal ring, $\exists f \in R[x]$ s.t. $\langle f \rangle = \langle r, x \rangle$

$\implies \exists p(x), q(x) \in R[x]$ s.t. $f(x)p(x) = r$ and $f(x)q(x) = x$

By looking at degrees, it follows that

$$f(x) = a \in R \qquad \text{and} \qquad q(x) = bx + c, \text{ where } b, c \in R$$

$$\begin{aligned}
\implies x &= f(x)q(x) \\
&= a(bx + c) \\
&= (ab)x + ac
\end{aligned}$$

$\implies ab = 1$

$\implies a$ is a unit

$\implies \langle f \rangle = \langle 1 \rangle$ since $\langle f \rangle = \langle a \rangle = \{ag(x) : g(x) \in R[x]\}$ and $ab = 1$

$\implies \exists s,t \in R[x]$ s.t. $rs + tx = 1$ since $\langle r, x \rangle = \langle f \rangle = 1$

$\implies s_0 r = 1$ where s_0 is the constant term of $s(x)$

$\implies r$ is invertible

Since r was arbitrary, R is closed under multiplicative inverses.
Thus, R is a field.