Rutgers University: Algebra Written Qualifying Exam

January 2017: Problem 1

Exercise. Prove that any complex square matrix is similar to its transpose matrix.

Solution.

Let A be a complex square matrix with Jordan form

$$J = \begin{bmatrix} J_{\lambda_1,k_1} & & 0 \\ 0 & \ddots & \\ & & J_{\lambda_m,k_m} \end{bmatrix} \qquad \text{then } \exists P \text{ s.t. } A = PJP^{-1}. \text{ And so, } A \sim J$$

Look at a Jordan block:

$$J_{\lambda_i,k_i} = \begin{bmatrix} \lambda_1 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \qquad \text{And let } B = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$$

Show: $J_{\lambda_i,k_i} \sim J^T_{\lambda_i,k_i}$. Then

$$B^{-1} = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \quad \text{and} \quad BJ_{\lambda_i,k_i}B^{-1} = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}\begin{bmatrix} \lambda_1 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}\begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & & 1 & \lambda_i \\ & \iddots & \iddots & \\ 1 & \iddots & 0 & \\ \lambda_i & & & \end{bmatrix}\begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & & 0 & \\ 1 & \ddots & & \\ 0 & \ddots & \ddots & \\ & & 1 & \lambda_i \end{bmatrix}$$

$$= J^T_{\lambda_i,k_i}$$

Looking at all of J again, if J is an $n \times n$ matrix and Q is the $n \times n$ matrix $\begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$, then

$$QJQ^{-1} = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}\begin{bmatrix} J_{\lambda_1,k_1} & & 0 \\ 0 & \ddots & \\ & & J_{\lambda_m,k_m} \end{bmatrix}\begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} = J^T$$

$$\begin{aligned} A^T = (PJP^{-1})^T = (P^{-1})^T J^T O^T &= (P^T)^{-1} J^T P^T \\ &= (P^T)^{-1}[QJQ^{-1}]P^T \\ &= (P^T)^{-1}Q[P^{-1}AP]Q^{-1}P^T \\ &= (PQ^{-1}P^T)^{-1}A(PQ^{-1}P^T) \qquad \Longrightarrow A \sim A^T \end{aligned}$$