Rutgers University: Real Variables and Elementary Point-Set Topology Qualifying Exam

January 2016: Problem 1 Solution

Exercise. A subset A of $\mathbb{R}^n$ is said to be *path-connected* if, given any two points $x_0, y_0 \in A$, there exists a continuous path $\phi : [0,1] \to A$ such that $\phi(0) = x_0$ and $\phi(1) = y_0$

(a) Prove that if $A \subset \mathbb{R}^n$ is non-empty and path-connected, then A is connected.

Solution.

A is **connected** if it cannot be written as the union of two disjoint nonempty open sets.
Assume A is disconnected. Then $\exists X, Y \subset A$ s.t.

i) $X \neq \emptyset$ and $Y \neq \emptyset$ are both open

ii) $A = X \cup Y$, and

iii) $X \cap Y = \emptyset$.

Since $A = X \cup Y$ is path-connected, for $x_0 \in X \subset A$ and $y_0 \in Y \subset A$, there exists a continuous path $\phi : [0,1] \to X \cup Y$ such that $\phi(0) = x_0$ and $\phi(1) = y_0$.
Thus,

i) $\phi^{-1}(X) \subset [0,1]$ and $\phi^{-1}(Y) \subset [0,1]$ are both open and non-empty

ii) $[0,1] = \phi^{-1}(X) \cup \phi^{-1}(Y)$

iii) $\phi^{-1}(X) \cap \phi^{-1}(Y) = \emptyset$

Therefore, $[0,1]$ disconnected. But $[0,1]$ is connected, so this gives us a contradiction!
Thus, A must be connected.

(b) Suppose now that A is an open subset of $\mathbb{R}^n$. For $x \in A$, let C_x be the set of points z in A for which there is a continuous path in A from x to z. Prove that C_x is open in A. (Hint: use the fact that every ball in $\mathbb{R}^n$ is path-connected, and use composition of paths.)

Solution.

Let $z \in C_x$.
Then exists a continuous path in A from x to z.
Since A is open, there exists a ball $B_z \subset A$ centered at z.
Let $z_0 \in B_z$.
Since every ball in $\mathbb{R}^n$ is path-connected, there exists a continuous path in A from z to z_0.
Using the composition of paths, it follows that there is a continuous path from x to z_0.
Thus, $z_0 \in C_x$, and so $B_z \subset C_x$ since $z_0 \in B_z$ was arbitrary.
Since $z \in C_x$ was arbitrary, it follows that C_x is open in A.

(c) Continuing with the assumptions of part (b), prove that for any two points $x, y \in A$ either $C_x = C_y$ or $C_x \cap C_y = \emptyset$.

Solution.

Let $x, y \in A$ and suppose that $C_x \cap C_y \neq \emptyset$.
Let $z \in C_x \cap C_y$ and $z_1 \in C_x$.
Then, there exists continuous paths connecting y to z, z to x, and x to z_1.
Using the composition of paths, it follows that there exists a continuous path from y to z_1.
Thus, $z_1 \in C_y$, and so $C_x \subseteq C_y$.

Similarly if $z_2 \in C_y$, then there exists continuous paths connecting x to z, z to y, and y to z_2
Using the composition of paths, it follows that there exists a continuous path from x to z_2.
Therefore, $z_2 \in C_x$, and so $C_y \subseteq C_x$.
Thus, we conclude that $C_x = C_y$.

(d) Continuing with the assumptions of parts (b) and (c), prove that if A is connected, then A is also path-connected. (Hint: use the fact that A can be written as $\bigcup\limits_{x \in A} C_x$)

Solution.

Suppose that A is connected and note that $A = \bigcup\limits_{x \in A} C_x$.

Let $x_0 \in A$ and suppose $C_{x_0} \subsetneq A$.

$\implies \exists y_0 \in A$ s.t. $y_0 \notin C_{x_0}$.

$\implies \bigcup\limits_{y \in A \setminus C_{x_0}} C_y \neq \emptyset$ and $C_{x_0} \neq \emptyset$.

Moreover, in part (b) we showed that C_{x_0} and C_y are both open for all y

$\implies \left(\bigcup\limits_{y \in A \setminus C_{x_0}} C_y \right)$ is open since it is the union of open sets.

Since $y \notin C_{x_0}$ for any y, it follows that $C_{x_0} \neq C_y$.
By part (c), $C_{x_0} \cap C_y = \emptyset$ for all $y \in A \setminus C_{x_0}$

$\implies C_{x_0} \cap \left(\bigcup\limits_{y \in A \setminus C_{x_0}} C_y \right) = \emptyset$. Thus,

$$A = C_{x_0} \cup \left(\bigcup_{y \in A \setminus C_{x_0}} C_y \right)$$

is the union of two disjoint nonempty open sets.
But then, A is not connected, so this is a contradiction!
Thus, $A = C_{x_0}$, which is path-connected.