# Rutgers University: Algebra Written Qualifying Exam

**January 2016: Problem 3 Solution**

**Exercise.** Let $A \in M_n(\mathbb{C})$ be a matrix such that $A^k = A$ for some integer $k \geq 2$. Prove that $A$ is diagonalizable.

**Solution.**

$A^k - A \implies A(A^{k-1} - I) = 0$

$\implies$ $\exists$a monic polynomial $p \in \mathbb{C}[x]$, $p(x) = x(x^{k-1} - 1)$ such that $p(A) = 0$

$p$ has $k$ distinct roots (namely 0 and the $(k-1)^{th}$ roots of unity.

i.e. $p$ has simple roots.

$A$ is diagonalizable $\iff$ $\exists p \in \mathbb{C}[x]$ such that $p(A) = 0$ AND $p$ has no repeated roots.

So, $A$ is diagonalizable.