Rutgers University: Algebra Written Qualifying Exam

January 2016: Problem 2 Solution

Exercise. Let M denote the additive group $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$ and let $\text{End}(M)$ denote the set of homomorphisms $\phi : M \to M$. Show that $\text{End}(M)$ is infinite and noncommutative.

Solution.

Let $\phi_k : M \to M$ be defined by $\phi_k((a,b)) = (ka, b)$. Then

$$\begin{aligned}
\phi_k((a,b)+(c,d)) &= \phi_k((a+c, b+d)) \\
&= (k(a+c), b+d) \\
&= (ka+kc, b+d) \\
&= (ka, b) + (kc, d) \\
&= \phi_k((a,b)) + \phi_k((c,d)) \\
&\implies \phi_k \in \text{End}(M)\ \forall k \in \mathbb{Z} \\
&\implies \text{End}(M) \text{ is infinite.}
\end{aligned}$$

Now let $\phi : M \to M$ be defined by $\phi((a,b)) = (a+b, b)$

$$\begin{aligned}
\phi((a,b)+(c,d)) &= \phi((a+c, b+d)) \\
&= (a+c+b+d, b+d) \\
&= (a+b, b) + (c+d, d) \\
&= \phi((a,b)) + \phi((c,d)) \\
&\implies \phi \in \text{End}(M)
\end{aligned}$$

Let $\psi : M \to M$ be defined by $\psi((a,b)) = (a, 0)$.

$$\begin{aligned}
\psi((a,b)+(c,d)) &= \psi((a+c, b+d)) \\
&= (a+c, 0) \\
&= (a,0) + (c,0) \\
&= \psi((a,b)) + \psi((c,d)) \\
&\implies \psi \in \text{End}(M)
\end{aligned}$$

$$\begin{aligned}
\text{But } \phi \circ \psi((2,1)) &= \phi(2,0) = (2,0) \\
\text{and } \psi \circ \phi((2,1)) &= \phi(3,1) = (3,0) \\
\implies \ \phi \circ \psi((2,1)) &\neq \psi \circ \phi((2,1)) \\
\implies \ \text{End}(M) &\text{ is noncommutative}
\end{aligned}$$