Rutgers University: Complex Variables and Advanced Calculus Written Qualifying Exam

January 2015: Problem 5 Solution

Exercise. Prove that if $u : \mathbb{C} \to \mathbb{R}$ is a harmonic function which is bounded below (i.e. there exists a real number C such that $u(z) \geq C$ for all $z \in \mathbb{C}$), then u must be constant.

Solution.

$$u \text{ is harmonic} \qquad \Longrightarrow \qquad u_{xx} + u_{yy} = 0$$

Theorem: A function is holomorphic IFF its real and imaginary parts are harmonic

If u is harmonic, $\exists v$ s.t. $f = u + iv$ is holomorphic
$\Longrightarrow$ e^{-u-iv} is analytic on $\mathbb{C}$ and

$$\begin{aligned} |e^{-u-iv}| &= |e^{-u}| \cdot |e^{-iv}| \\ &\leq |e^{-u}| \\ &= e^{-u} \\ &< e^{-C}, & \text{since } -C > -u(z) \text{ for all } z \in \mathbb{C} \end{aligned}$$

$\Longrightarrow$ e^{-u-iv} is entire and bounded.
By **Louiville's Theorem**, e^{-u-iv} is constant.
$\Longrightarrow$ $-u - iv$ is constant
$\Longrightarrow$ u is constant