# Rutgers University: Algebra Written Qualifying Exam
## January 2015: Problem 5 Solution

**Exercise.** Let $\rho : G \to GL_3(\mathbb{C})$ be a homomorphism, where $G$ is the cyclic group of order 3. Show that with respect to some basis of $\mathbb{C}^3$, every element of $\rho(G)$ is a diagonal matrix having cube roots of unity on its diagonal.

**Solution.**

Suppose $G = \{e, a, a^2\}$ and $\rho : G \to GL_3(\mathbb{C})$ is a homomorphism.

$$\rho(e) = I_3$$
$$\rho(a^2) = \rho(a)\rho(a)$$
$$\rho(a) = \rho(a^2 \cdot a^2) = \rho(a^2)\rho(a^2) = [\rho(a)]^4$$
$$I_3 = \rho(e) = \rho(a^3) = \rho(a)\rho(a^2) = [\rho(a)]^3 = [\rho(a^2)]^3$$

If $\rho(a) = A$ then $$A^3 = [\rho(a)]^3 = \rho(a^3) = \rho(e) = I_3$$

$\implies$ $$A^3 - I_3 = 0$$

$\implies$ $$p_A(x) = (x-q)(x-\omega)(x-\omega^2) \text{ where } \omega \text{ is the cube root of unity}$$

Looking at the Jordan canonical form of $A$, $A = PJP^{-1}$ has eigenvalues $1, \omega, \omega^2$

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}$$

$A^2 = PJ^2P^{-1}$ and $$J^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix}$$

and $J^3 = I_3$ $\implies$ $PJ^3P^{-1} = I_3$

So, $\rho(e) = PI_3P^{-1}$ $\quad \rho(a) = PJP^{-1}$ $\quad \rho(a^2) = PJ^2P^{-1}$

Similar matrices represent the same matrix under 2 bases

$\implies \rho(e) = I_3 \qquad \rho(a) = J \qquad \rho(a^2) = J^2 \qquad$ with respect to some basis of $\mathbb{C}^3$

$\implies$ The elements of $\rho(G)$ are diagonal matrices having cube roots of unity on its diagonal.