Rutgers University: Algebra Written Qualifying Exam
January 2015: Problem 4 Solution

Exercise. Recall that the group $GL_2(\mathbb{R})$ acts on $\mathbb{R}^2$ by the usual matrix-vector multiplication $A \cdot v = Av$, where $A \in GL_2(\mathbb{R})$ and v is a column vector in $\mathbb{R}^2$.

(a) Determine the number of orbits for this action, and describe each orbit

Solution.

Suppose a group G acts on set X. For $x \in X$, the **orbit** of x is

$$orb(x) = \{gx : g \in G\} \subseteq X$$

So, the number of orbits = number of x's s.t. $\{gx : g \in G\}$ is different

Look at $\vec{v} = \vec{0}$:

$orb(\vec{0}) = \{A \cdot \vec{0} : A \in GL_2(\mathbb{R})\} = \{0\}$

So this is one orbit.

Look at $\vec{v} \neq \vec{0}$:

$orb(\vec{v}) = \{A\vec{v} : A \in GL_2(\mathbb{R})\}$

If $\vec{v} \neq \vec{0}$, $\exists \vec{u} \neq \vec{0}$ s.t. $\{\vec{v}, \vec{w}\}$ forms a basis of $\mathbb{R}^2$

$\vec{v} \in orb(\vec{v})$ since $I\vec{v} = \vec{v}$

$$v \in orb\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \text{ since } \begin{bmatrix} v_1 & u_1 \\ v_2 & u_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{v}$$

$$\implies orb\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \left\{\vec{v} \in \mathbb{R}^2 \backslash \{(0,0)\}\right\}$$

Also, for $\vec{w} \in \mathbb{R}^2 \backslash \{(0,0)\}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ where $w_1, w_2 \in \mathbb{R}$ and not both 0

If $w_1 \neq 0$ and $w_2 = 0$ then $\begin{bmatrix} v_1/w_1 & u_1 \\ v_2/w_1 & u_2 \end{bmatrix} \begin{bmatrix} w_1 \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{v}$

If $w_1 = 0$ and $w_2 \neq 0$ then $\begin{bmatrix} u_1 & v_1/w_2 \\ u_2 & v_2/w_2 \end{bmatrix} \begin{bmatrix} 0 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{v}$

If $w_1 \neq 0$ and $w_2 \neq 0$ and $v_1 \neq 0$ and $v_2 \neq 0$ then $\begin{bmatrix} v_1/w_1 & 0 \\ 0 & v_2/w_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{v}$

If $w_1 \neq 0$ and $w_2 \neq 0$ AND EITHER $v_1 = 0$ and $v_2 \neq 0$ OR $v_1 \neq 0$ and $v_2 = 0$ then

$$\begin{bmatrix} \frac{v_1}{2w_1} & \frac{v_1}{2w_2} \\ \frac{v_2}{2w_1} & \frac{v_2}{2w_2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{v}$$

So $\vec{v} \in orb(\vec{w})$ for all $\vec{w} \neq \vec{0}$, but $\vec{v} \neq \vec{0}$ was arbitrary

$\implies \forall \vec{w} \neq \vec{0}$, $orb(\vec{w}) = \mathbb{R} \backslash \{(0,0)\}$

There are two orbits: $orb(\vec{0}) = \{\vec{0}\}$ and $orb(\vec{w}) = \mathbb{R} \backslash \{(0,0)\}$ for $\vec{w} \neq \vec{0}$

(b) Find the pointwise stabilizer of the set $\{(x, y) \in \mathbb{R}^2 | y = x, x \neq 0\}$

Solution.

The **stabilizer** of x is

$$stab(x) = \{g \in G : gx = x\}$$

For $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R})$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} ax + bx \\ cx + dx \end{bmatrix} = \begin{bmatrix} (a+b)x \\ (c+d)x \end{bmatrix}$$

So for $\begin{bmatrix} x \\ x \end{bmatrix}$ where $x \neq 0$,

$$\begin{aligned} stab\left(\begin{bmatrix} x \\ x \end{bmatrix}\right) &= \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R}) : \begin{bmatrix} (a+b)x \\ (c+d)x \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix}\right\} \\ &= \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R}) : \begin{matrix} a+b=1 \\ c+d=1 \end{matrix}\right\} \\ &= \left\{\text{invertible matrices } \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{matrix} a+b=1 \\ c+d=1 \end{matrix}\right\} \end{aligned}$$