Rutgers University: Algebra Written Qualifying Exam
January 2015: Problem 3 Solution

Exercise. Prove that a finite group G is the internal direct product of its Sylow subgroups if and only if every Sylow subgroup is normal in G.

Solution.

Suppose $|G| = \prod_{i=1}^{n} p_i^{\alpha_i}$.

($\Longleftarrow$) If every Sylow subgroup is normal in G, then for every p, the $p-$Sylow subgroup is unique

(By the third Sylow theorem: $n_p = 1 \iff$ Sylow $p-$ subgroup is a normal subgroup)

$\implies$ The Sylow subgroups are $P_1, \ldots, P_n$ where $|P_i| = p_i^{\alpha_i}$, each P_i has unique order, and

$$P_i \cap P_j = \{e\} \text{ for all } i, j = 1, \ldots, n \text{ where } i \neq j$$
$$|P_1 \ldots P_n| = \frac{|P_1| \ldots |P_n|}{1} = \prod_{i=1}^{n} p_i^{\alpha_i} = |G|$$

Moreover, $P_1 \ldots P_n \subseteq G$ since $P_i \subseteq G$ and G is closed

$\implies P_1 \ldots P_n = G$

Thus, G is the internal direct product of its Sylow groups.

($\Longrightarrow$) Suppose G is the internal direct product of its Sylow subgroups, denoted H_i for $i = 1, \ldots, m$

Let $\phi : H_1 \times \cdots \times H_m \to G$ be defined by $\phi((h_1, \ldots, h_m)) = h_1 \ldots h_m$

ϕ is an isomorphism, and

$$\phi((e, \ldots, e, h_{i1}, e, \ldots, e, h_{j1}, e, \ldots, e)(e, \ldots, e, h_{i2}, e, \ldots, e, h_{j2}, e, \ldots, e))$$
$$= \phi((e, \ldots, e, h_{i1}, e, \ldots, e, h_{j1}, e, \ldots, e))\phi((e, \ldots, e, h_{i2}, e, \ldots, e, h_{j2}, e, \ldots, e))$$
$$\phi((e, \ldots, e, h_{i1}h_{i2}, e, \ldots, e, h_{j1}h_{j2}, e, \ldots, e))$$
$$= e \ldots e h_{i1} e \ldots e h_{j1} e \ldots e h_{i2} e \ldots e h_{j2} e \ldots e$$
$$h_{i1}h_{i2}h_{j1}h_{j2} = h_{i1}h_{j1}h_{i2}h_{j2}$$
$$h_{i2}h_{j1} = h_{j1}h_{i2}$$

$\implies$ The elements of H_i commute with the elements of H_j for $i \neq j$.

Since ϕ is an isomorphism, it is surjective

So, $\forall a \in G$, $\exists h_i \in H_i$ for $i = 1, \ldots, m$ s.t. $a = \prod_{i=1}^{m} h_i$

So for $h \in H_k$ where $1 \leq k \leq m$

$$aha^{-1} = (h_1 \ldots h_m)h(h_1 \ldots h_m)^{-1}$$
$$= h_1 \ldots h_m h h_m^{-1} \ldots h_1^{-1}$$
$$= h_k h h_k^{-1} h_1 h_1^{-1} \ldots h_{k-1}h_{k-1}^{-1}h_{k+1}h_{k+1}^{-1} \ldots h_m h_m^{-1} \text{ by comm. between } H_i \text{ and } H_j i \neq j$$
$$= h_k h h_k^{-1} \in H_k$$

So $\forall a \in G$ $aha^{-1} \in H_k$. Since h is arbitrary, $H_k \lhd G$

Since k was arbitrary, this holds for all k,

In other words, every Sylow subgroup is normal in G