Rutgers University: Algebra Written Qualifying Exam

January 2015: Problem 2 Solution

Exercise. Let $\mathbb{Z}[x]$ denote the polynomial ring in the variable x with coefficients in $\mathbb{Z}$.

(a) Let $I \subset \mathbb{Z}[x]$ be the ideal consisting of all elements whose constant term is 0. Prove that I is a prime ideal of $\mathbb{Z}[x]$ but not a maximal ideal.

Solution.

I is an **ideal** so $(I, +)$ is a group and $\forall r \in \mathbb{Z}[x], i \in I$,

$$ri \in I \text{ and } ir \in I.$$

Prime: An ideal I is **prime** if $ab \in I \implies a \in I$ or $b \in I$.

Let $a, b \in \mathbb{Z}[x]$. Then

$$a(x) = a_n x^n + \cdots + a_1 x + a_0 \qquad \text{and} \qquad b(x) = b_m x^m + \cdots + b_1 x + b_0.$$

The constant term of ab is $a_0 b_0$.
Therefore, if $ab \in I$, then $a_0 b_0 = 0$
$\implies a_0 = 0$ or $b_0 = 0$
$\implies a \in I$ or $b \in I$
Thus, I is a prime ideal in $\mathbb{Z}[x]$.

Not Maximal: An ideal I is called **maximal** if $\nexists$ an ideal J s.t. $I \subsetneq J \subsetneq \mathbb{Z}[x]$

We want to find an ideal J s.t. $I \subsetneq J \subsetneq \mathbb{Z}[x]$
Let J consist of all elements with an even constant term
Then $I \subsetneq J \subsetneq \mathbb{Z}[x]$
J is an ideal because for $a \in J$ and $b \in \mathbb{Z}[x]$,
ab and ba have even constant term $a_0 b_0$, since a_0 is even and $b_0 \in \mathbb{Z}$
and $(J, +)$ is closed under addition and inverses and contains the identity 0.

(b) Prove that $\mathbb{Z}[x]$ is not a principal ideal domain

Solution.

A **principal ideal domain** is an integral domain (i.e. commutative ring with multiplicative identity and no zero divisors) in which every proper ideal can be generated by a single element.

J, as defined in part (a), is an ideal that is **NOT** principal.
$5x + 2 \in J$ but it cannot be reduced in $\mathbb{Z}[x]$
$\implies$ If $\mathbb{Z}[x]$ is a PID, then J must be a principal ideal and generated $5x + 2$.
But $x + 2 \in K$ and $x + 2 \notin \langle 5x + 2 \rangle$.
$\implies J$ is not a principal ideal
$\implies \mathbb{Z}[x]$ is not a PID.