Rutgers University: Real Variables and Elementary Point-Set Topology Qualifying Exam

January 2008 Day 2: Problem 4 Solution

Exercise. Let f be a Lebesgue integrable function on $\mathbb{R}$. Prove that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}(\cos x)^n f(x)dx = 0$$

Solution.

Dominated Comvergence Theorem: Let $\{f_n\}$ be a sequence in L^1 s.t.

(a) $f_n \to f$ a.e. and

(b) $\exists$a non-negative $g \in L^1$ s.t. $|f_n| \le g$ a.e. $\forall n$

Then $f \in L^1$ and $\int f = \lim_{n\to\infty}\int f_n$

Let
$$h_n(x) = (\cos x)^n f(x)$$
$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty}(\cos x)^n f(x) = 0$$

and since $|cosx| \le 1$,
$$|(\cos x)^n| \le 1$$

$\implies$
$$\begin{aligned}|h_n(x)| &= |(\cos x)^n f(x)| \\ &= |(\cos x)^n||f(x)| \\ &\le |f(x)|\end{aligned}$$

and since $\displaystyle\int |f(x)|dx < \infty$,
$$|f| \in L^1$$

Thus, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int(\cos x)^n f(x)dx = \int_{\mathbb{R}} 0dx = 0$$