Rutgers University: Algebra Written Qualifying Exam

January 2012: Day 2 Problem 1 Solution

Exercise. Suppose A is a 5×5 complex matrix and $(A - 2I)^5 = 0$.

(a) What Jordan canonical forms are possible for A?

Solution.

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix},$$

(Part b is on the next page)

(b) Suppose that there exists another 5×5 complex matrix B such that $AB = BA$ and the minimal polynomial of B is $t^3 + t$. Now which of your answers to (a) are still possible Jordan canonical forms for A? Explain your reasoning.

Solution.

$q_B(t) = t(t-i)(t+i)$

and B is diagonalizable since $q_B(B) = 0$ and q_B has simple roots

Theorem: If $AB = BA$ then for any $\lambda \in \mathbb{C}, k \in \mathbb{Z}_{\geq 0}$, A sends $\ker((B - \lambda I)^k)$ to itself.

$$\left.\begin{aligned} p_B(t) = & t^3(t-i)(t+i), t(t-i)^3(t+i), t(t-i)(t+i)^3 \\ & t^2(t-i)^2(t+i), t^2(t-i)(t+i)^2, t(t-i)^2(t+i)^2 \end{aligned}\right\} \begin{array}{l} \text{Think of as:} \\ (x-r_1)^3(x-r_2)(x-r_3) \\ (x-r_1)^2(x-r_2)^2(x-r_3) \end{array}$$

Let $W_1 = \ker(B - r_1 I)$ $\quad W_2 = \ker(B - r_2 I)$ $\quad W_3 = \ker(B - r_3 I)$,

Then $V = \underbrace{W_1}_{\dim 2} \oplus \underbrace{W_2}_{\dim 2} \oplus \underbrace{W_3}_{\dim 1}$

Let $A_i = A|_{w_i}$ for $i = 1, 2, 3$

Since $A_1 = W_1 \to W_1$ is a 2×2 matrix. it has JCF $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ or $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

Similarly, $A_2 \sim \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} or \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$A_3 \sim \begin{bmatrix} 2 \end{bmatrix}$

The Jordan Canonical form of A is blocks $\begin{bmatrix} J_1 & & 0 \\ & J_2 & \\ 0 & & J_3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$