Rutgers University: Algebra Written Qualifying Exam

January 2008: Day 1 Problem 7 Solution

Exercise. Let $\mathbb{Z}^4 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ (also denoted $\mathbb{Z}^{(4)}$). Consider $\mathbb{Z}^4$ as an abelian group under addition of coordinates. Let S be the subgroup of $\mathbb{Z}^4$ generated by the elements

$$(5, -2, -4, 1), \qquad (-5, 4, 4, 1), \qquad (0, 6, 0, 6).$$

Determine the structure of the abelian group $\mathbb{Z}^4/S$ as a direct product of cyclic groups.

Solution.

Define $f : \mathbb{Z}^4 \to \mathbb{Z}^4$ and $S = \ker(f)$.
Goal: nice generators for f.
Let

$$M = \begin{bmatrix} 5 & -2 & -4 & 1 \\ -5 & 4 & 4 & 1 \\ 0 & 6 & 0 & 6 \end{bmatrix}$$

Allowed Operations:

- add integer multiple of one row or column to another
- swap rows or columns
- multiply row or column by -1 $\leftarrow$ since only ± 1 is invertible in $(\mathbb{Z}, \cdot)$

$$\begin{bmatrix} 5 & -2 & -4 & 1 \\ -5 & 4 & 4 & 1 \\ 0 & 6 & 0 & 6 \end{bmatrix} \overset{R_2+R_1\to R_2}{\sim} \begin{bmatrix} 5 & -2 & -4 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 6 & 0 & 6 \end{bmatrix} \overset{C_1+C_3\to C_1}{\sim} \begin{bmatrix} 1 & -2 & -4 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 6 & 0 & 6 \end{bmatrix}$$

$$\overset{2C_1+C_2\to C_2}{\sim} \begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 6 & 0 & 6 \end{bmatrix} \overset{C_4-C_1\to C_4}{\sim} \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 6 & 0 & 6 \end{bmatrix} \overset{R_3-2R_3\to R_3}{\sim} \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{C_3+4C_1\to C_3}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{C_4-2C_2\to C_4}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\mathbb{Z}_1 \times \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}}$$