# Rutgers University: Algebra Written Qualifying Exam

**January 2006: Day 1 Problem 1 Solution**

**Exercise.** How many different Sylow 2-subgroups and Sylow 5-subgroups are there in a non-commutative group of order 20? Explain your answer.

Solution.

Let $G$ be a group of order 20.

$$20 = 2^2 \cdot 5$$

By the third Sylow theorem,

$$\begin{array}{lllll} n_2 \equiv 1 \mod 2 & \text{and} & n_2 \mid 5 & \implies & n_3 = 1 \text{ or } 5 \\ n_5 \equiv 1 \mod 5 & \text{and} & n_5 \mid 4 & \implies & n_5 = 1 \end{array}$$

If $n_2 = 1$ then $\exists P_2, P_5$ and $P_2, P_5$ both normal.

$$\implies G = P_2 P_5 = P_5 P_2 \cong P_5 \times P_2$$

where $|P_5| = 5 \implies P_5$ cyclic $\implies P_5$ abelian
and $|P_2| = 4 \implies P_2 = \{e, x, y, z\}$.
If $P_2$ has an eleemnt of order 4 then $P_2$ is cyclic and therefore abelian
Otherwise, $x, y, z$ must have order 2
But then $xy \neq x, y,$ or $e$ by cancellation laws
$\implies xy = z$
Similarly , $yx = z$.
$\implies xy = yx$ and $P_2$ is abelian.
But then $G \cong P_5 \times P_2$ must also be abelian.
Since we were given that $G$ is not abelian, this is a contradiction!
Thus, $n_2 \neq 1$ and so $n_2 = 5$.

In other words, $G$ has one Sylow 5-subgroup and 5 Sylow 2-subgroups.