Rutgers University: Algebra Written Qualifying Exam

August 2013: Day 1 Problem 5 Solution

Exercise. Suppose that A is a square complex matrix and f is a polynomial in $\mathbb{C}[t]$ such that $f(A)$ is diagonalizable. If $f'(A)$ is invertible, where f' is the derivative of f, prove that A is diagonalizable in $\mathbb{C}$.

Solution.

$$f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$
$$f'(t) = na_n t^{n-1} + (n-1)a_{n-1}t^{n-2} + \cdots + a_1$$
$$f(A) = a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I \qquad \text{is diagonalizable.}$$

Recall: M is diagonalizable $\iff$ $\exists$a polynomial $q(t)$ with simple roots such that $q(A) = 0$

$\implies$ $\exists q(t) \in \mathbb{C}[t]$ s.t. $\quad q(t) = (t-b_1)(t-b_2)\dots(t-b_m) \quad b_i \neq b_j$ for $i \neq j$

and $\quad (f(A) - b_1 I)(f(A) - b_2 I)\dots(f(A) - b_m I) = 0$

$$f'(A) = na_n A^{n-1} + (n-1)a_{n-1}A^{n-2} + \cdots + a_1 I \qquad \text{invertible}$$

$\implies$ $\det(f'(A)) = c \neq 0$

Look at Jordan form! Let

$$A = PJP^{-1} \qquad \text{and} \qquad J = \begin{bmatrix} J_{\lambda_1,k_1} & & 0 \\ 0 & \ddots & \\ & & J_{\lambda_m,k_m} \end{bmatrix}$$

Look at a single Jordan block.
Suppose A is not diagonalizable. Then $\exists$a Jordan block J_{λ_i,k_i} of size greater than 1.

$$J_{\lambda_i,k_i} = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix} \qquad \text{and} \qquad J^{\ell}_{\lambda_i,k_i} = \begin{bmatrix} \lambda_i^{\ell} & \ell\lambda_i^{\ell-1} & & ? \\ & \ddots & \ddots & \\ & & \ddots & \ell\lambda_i^{\ell-1} \\ 0 & & & \lambda_i^{\ell} \end{bmatrix}$$

$$\begin{aligned} \implies f(A_i) &= a_n(PJP^{-1})^n + \cdots + a_1(PJP^{-1}) + a_0 I \\ &= P(a_n J^n + \cdots + a_1 J + a_0 I)P^{-1} \\ &= P \begin{bmatrix} f(\lambda) & f'(\lambda) & & ? \\ & \ddots & \ddots & \\ & & \ddots & f'(\lambda) \\ 0 & & & f(\lambda) \end{bmatrix} P^{-1} \end{aligned} \qquad \text{and} \qquad \det(f'(A)) \neq 0 \implies f'(\lambda) \neq 0$$

But since $f(A)$ is diagonalizable, its Jordan blocks have size 1.

$$\implies \quad f(A_i) = B \begin{bmatrix} f(\lambda) & & 0 \\ 0 & \ddots & \\ & & f(\lambda) \end{bmatrix} B^{-1} \qquad \text{for some } B, \text{ a contradiction!}$$

Thus, A is diagonalizable.