Rutgers University: Algebra Written Qualifying Exam

August 2010: Day 1 Problem 1 Solution

Exercise. Prove that no group of order 3400 is simple.

Solution.

Let G be a group of order 3400. We want to show that there is a normal subgroup of G that is ***not*** $\{e\}$ or G.

$$3400 = 2^3 \cdot 5^2 \cdot 17$$

By the third Sylow theorem,

$$\begin{array}{lll} & n_{17} \mid (2^3 \cdot 5^2) \quad \Longrightarrow & n_{17} = 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 200 \\ \text{and} & n_{17} \equiv 1 \mod 17 \quad \Longrightarrow & n_{17} = 1, 18, 35, 52, 69, 85, 102, \ldots, 187, 204 \\ \Longrightarrow & n_{17} = 1 & \end{array}$$

Since the number of 17−Sylow subgroups is $n_{17} = 1$, the 17−Sylow subgroup is a normal subgroup of G by the Second Sylow Theorem.
Thus, G is not simple.