Rutgers University: Real Variables and Elementary Point-Set Topology Qualifying Exam

August 2018: Problem 3 Solution

Exercise. Let $\pi(x, y) = x$ denote the projection of $\mathbb{R}^2$ onto $\mathbb{R}$, and let $\pi(A)$ denote the image under π of a subset of A of $\mathbb{R}^2$.

(a) Let μ^* be an outer measure on the subsets of $\mathbb{R}$. Show that $\nu^*(A) := \mu^*(\pi(A))$ is an outer measure on the subsets of $\mathbb{R}^2$.

Solution.

ν^* is an **outer measure** if

- $\nu^*(A) \geq 0$ for $A \subseteq \mathbb{R}^2$
- $\nu(\emptyset) = 0$
- $\nu^*(A) \leq \nu^*(B)$ if $A \subseteq B$
- $\nu^*(\cup_1^\infty A_j) \leq \sum_1^\infty \nu^*(A_j)$

- Let $A \subset \mathbb{R}^2$. Then $\nu^*(A) = \mu^*(\pi(A)) \geq 0$, since μ^* an outer measure on $\mathbb{R}$ and $\pi(A) \subseteq \mathbb{R}$.
- $\nu^*(\emptyset) = \mu^*(\pi(\emptyset)) = \mu^*(\emptyset) = 0$
- If $A \subseteq B$ then $\pi(A) \subseteq \pi(B)$ since

$$\begin{aligned} x \in \pi(A) &\implies (x, y) \in A \text{ for some } y \in \mathbb{R} \\ &\implies (x, y) \in B \\ &\implies x \in \pi(B) \end{aligned}$$

- $$\begin{aligned} \nu^*(A) = \mu^*(\pi(A)) &\leq \mu^*(\pi(B)) \qquad \text{since } \pi(A) \subset \pi(B) \text{ and } \mu^* \text{ an outer measure} \\ &= \nu^*(B) \end{aligned}$$

- $$\begin{aligned} \nu^*(\cup_1^\infty A_j) &= \mu^*(\pi(\cup_1^\infty A_j)) \\ &= \mu^*(\cup_1^\infty \pi(A_j)) \\ &= \sum_1^\infty \mu^*(\pi(A_j)) \\ &= \sum_1^\infty \nu^*(A_j) \end{aligned}$$

Thus, ν^* is an outer measure on $\mathbb{R}^2$.

(b) Let λ^* be Lebesgue outer measure on the subsets of $\mathbb{R}$, and let $\rho^*(A) = \lambda^*(\pi(A))$. Show that if $A = B \times \mathbb{R}$, where B is a Lebesgue measurable subset of $\mathbb{R}$, then A is a ρ^* measurable set. Show where the assumption that A has this particular form is used.

Solution.

A is **ρ^*−measurable** iff $\rho^*(E) = \rho^*(E \cap A) + \rho^*(E \cap A^C$ for all $E \subset X$.

Let $A = B \times \mathbb{R}$, where B is Lebesgue measurable subset of $\mathbb{R}$, and let $E \subseteq \mathbb{R}^2$, $E = E_1 \times E_2$.

$$\begin{aligned}
\rho^*(E) = \lambda^*(\pi(E)) &= \lambda^*(E_1) \\
\rho^*(E \cap A) + \rho^*(E \cap A^C) &= \lambda^*(\pi(E \cap A)) + \lambda^*(\pi(E \cap A^C)) \\
&= \lambda^*(\pi((E_1 \times E_2) \cap (B \times \mathbb{R}))) + \lambda^*(\pi((E_1 \times E_2) \cap (B \times \mathbb{R})^C)) \\
&= \lambda^*(E_1 \cap B) + \lambda^*(E_1 \cap B^C) \\
&= \lambda^*(E_1) \\
&= \rho^*(E)
\end{aligned}$$