Rutgers University: Complex Variables and Advanced Calculus Written Qualifying Exam

August 2018: Problem 4 Solution

Exercise.

(a) Suppose that U is a domain in the complex plane, that f is holomorphic in U and continuous over $\overline{U}$, and that $|f(z)| = 1$ for all $z \in \delta U$. Prove that, unless f is a constant function, $f(z) = 0$ must have a solution in U.

Solution.

Suppose $f(z) \neq 0$ for all $z \in U$.
Then by the **minimum modulus theorem** f is holomorphic over bounded domain U, continuous over $\overline{U}$, and nonzero at all points, so f attains its minimum on δU

$\implies |f(z)| \geq 1$ for all $z \in U$

$\implies \exists z_0 \in U$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in \overline{U}$.

Therefore, f is holomorphic on U and $|f|$ attains its maximum in U, so by the **maximum modulus principle** f is constant.
Thus, if f is nonconstant, $f(z) = 0$ must have a solution in U.

(b) Suppose that f is an entire function, and that $|f(z)| = 1$ for all $z \in \mathbb{C}$ with $|z| = 1$. Prove that $f(z) = az^n$ for some $a \in \mathbb{C}$ with $|a| = 1$ and $n \in \mathbb{Z}_{\geq 0}$. (Hint: *Note that the family* $\phi_c(z) := (z - c)/(1 - \overline{c}z$, *for* $|z| < 1$, *is meromorphinc on* $\mathbb{C}$, *satisfies* $|\phi_c(z)| = 1$ *for* $|z| = 1$, *and* $\phi_c(1/\overline{z})\phi_c(z) = 1$. *You may want to study* $f(z)$ *in relation to this family.*

Solution.

Look at power series expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

If $a_n = 0$ for all n then $f(z) \equiv 0$, a contradiction.
Choose k such that $a_n = 0$ for $n < k$ but $a_k \neq 0$.

$$\implies \qquad f(z) = \sum_{n=k}^{\infty} a_n z^n = z^k \sum_{n=0}^{\infty} a_{n+k} z^n$$

Let $g(z) = \sum_{n=0}^{\infty} a_{n+k} z^n$.

g is continuous and $g(0) = a_k \neq 0$

$\implies \exists r > 0$ s.t. $|g(z) - g(0)| < |a_k|$ for $|z| < r$

$\implies g(z) \neq 0$ on $D_r(0)$

By part (a), $g(z)$ is constant on $D_r(0)$, and so $f(z) = z^k g(z) = az^k$ for some $a \in \mathbb{C}$ and $|f(z)| = 1$ when $|z| = 1$.
Therefore, $|a| = 1$, amd $f(z) = az^n$ on $D_r(0)$, as desired.