Rutgers University: Algebra Written Qualifying Exam

August 2018: Problem 5 Solution

Exercise. Let G be a finite subgroup of the group of real $n \times n$ matrices with nonzero determinant such that all elements of G are symmetric matrices. Prove that G is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$ for some $k \geq 0$.

Solution.

A is **symmetric** if $A = A^T$.

Spectral Theorem for Symmetric Matrices: Let A be an $n \times n$ symmetric matrix over $\mathbb{R}$

- Every eigenvalue of A is real
- $\exists$diagonal matrix D and orthogonal matrix $(U^T = U^{-1})$ U in $M_n(\mathbb{R})$ such that

$$A = UDU^T = UDU^{-1}$$

Aside

If G is a finite group and $g^2 = 1$ for all $g \in G$, then $G \cong (\mathbb{Z}/2\mathbb{Z})^k$ for some k.

(a) If $g^2 = 1$ for all g then G is abelian

$$\begin{aligned} & (xy)^2 = 1 = (xy)(xy)^{-1} \\ \implies \quad & xyxy = xyy^{-1}x^{-1} \\ \implies \quad & xy = y^{-1}x^{-1} \\ \implies \quad & xy = yx \end{aligned}$$

(b) Since G is abelian and the order of every (non-identity) element is 2, we have $G \cong (\mathbb{Z}/2\mathbb{Z})^k$

Let G be a finite subgroup of $GL_n(\mathbb{R})$ s.t. every $A \in G$ is symmetric. Let $A \in G$.
Since G is finite, A has finite order.
$\implies$ $A^m = I$ for some $m \geq 1$.

Show: $A^2 = I$. Using Spectral theorem, decompose
$A = UDU^{-1}$ where diagonal entries of D are real and eigenvalues of A.
Since $A^m = I$ for some m, we have $\lambda^m = 1$ for any eigen valye of A
$\implies$ the eigenvalues of A are all ± 1
$\implies$ D is a diagonal matrix with diagonal entries ± 1. In particular, $D^2 = 0$.

$$A^2 = (UDU^{-1})(UDU^{-1}) = UD^2U^{-1} = UIU^{-1} = I$$

So, by the aside $G \cong (\mathbb{Z}/2)^k$ for some $k \geq 0$.