Rutgers University: Algebra Written Qualifying Exam

August 2018: Problem 4 Solution

Exercise. Let $G = \mathbb{Q}/\mathbb{Z}$ be the quotient of the additive group of rational numbers by the subgroup of integers.

(a) Prove that every finitely generated subgroup G is a finite cyclic subgroup.

Solution.

$$\mathbb{Q}/\mathbb{Z} = \{\frac{m}{n} + \mathbb{Z} : m, n \in \mathbb{Z}, \gcd(m, n) = 1 \text{ or } \frac{m}{n} = 0\}.$$

Let H be a finitely generated subgroup of G with generators

$$\frac{m_1}{n_1} + \mathbb{Z}, \ldots, \frac{m_k}{n_k} + \mathbb{Z}.$$

Then $H \subseteq \left\langle \dfrac{1}{n_1 n_2 \ldots n_k} + \mathbb{Z} \right\rangle$

Since H is a subgroup of a cyclic group, H is cyclic.

Also, $|H| \leq \left| \frac{1}{n_1 n_2 \ldots n_k} + \mathbb{Z} \right| = n_1 n_2 \ldots n_k$

Thus, H is a finite cyclic subgroup.

(b) Prove that G is not isomorphic to $G \oplus G$ as an abelian group.

Solution.

By part (a), every finitely generated subgroup of G is cyclic.

It suffices to find a finitely generated subgroup of $G \oplus G$ that is not cyclic.

Let $H = \langle \frac{1}{4} + \mathbb{Z} \rangle \oplus \langle \frac{1}{6} + \mathbb{Z} \rangle$.

H is a subgroup of $G \oplus G$ since $\langle \frac{1}{4} + \mathbb{Z} \rangle$ and $\langle \frac{1}{6} + \mathbb{Z} \rangle$ are both subgroups of $G = \mathbb{Q}/\mathbb{Z}$.

But H is not cyclic since:

$|\langle \frac{1}{4} + \mathbb{Z} \rangle| = 4$ and $|\langle \frac{1}{6} + \mathbb{Z} \rangle| = 6$

$\implies |H| = \left| \langle \frac{1}{4} + \mathbb{Z} \rangle \oplus \langle \frac{1}{6} + \mathbb{Z} \rangle \right| = |\langle \frac{1}{4} + \mathbb{Z} \rangle| \cdot |\langle \frac{1}{6} + \mathbb{Z} \rangle| = 24$

but for any $h \in H$ $o(h) \leq lcm(4, 6) = 12 < 24 = |H|$,

$\implies$ no element h can generated the whole subgroup H.

$\implies$ H is a finitely generated subgroup of $G \oplus G$ that is not cyclic.

$\implies$ G is not isomorphic to $G \oplus G$ as an abelian group.