Rutgers University: Real Variables and Elementary Point-Set Topology Qualifying Exam

August 2017: Problem 1 Solution

Exercise. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L^1(X, \mathcal{M}, \mu)$ be such that $f(x) > 0$ almost everywhere. Let $E \in \mathcal{M}$ be such that

$$\int_E f d\mu < \infty.$$

Prove that

$$\lim_{k\to+\infty} \int_E f^{\frac{1}{k}} d\mu(x) = \mu(E).$$

Solution.

$$\begin{aligned} \int_E f d\mu < \infty \quad &\implies \quad f(x) < \infty \quad &&\text{for almost every } x \in E \\ &\implies \quad \lim_{k\to+\infty} f^{\frac{1}{k}}(x) = 1 \quad &&\text{for almost every } x \in E \end{aligned}$$

Let $E_1 = \{x \in E : f(x) \leq 1\}$ and $E_2 = \{x \in E : f(x) > 1\}$.

$$\begin{aligned} &\implies \quad E = E_1 \cup E_2 \quad \text{and} \quad E_1 \cap E_2 = \emptyset \\ &\implies \quad \int_E f^{\frac{1}{k}} d\mu = \int_{E_1} f^{\frac{1}{k}} d\mu + \int_{E_2} f^{\frac{1}{k}} d\mu \end{aligned}$$

Note: $\forall x \in E_1$ and $k, \ell \in \mathbb{Z}^+$,

$$\begin{aligned} f^{\frac{1}{k}}(x) &\leq f^{\frac{1}{k+\ell}}(x) && \text{since } f(x) \leq 1 \\ \implies \lim_{k\to+\infty} \int_{E_1} f^{\frac{1}{k}} d\mu &= \int_{E_1} \lim_{k\to+\infty} f^{\frac{1}{k}} d\mu && \text{by the Monotone Convergence Theorem} \\ &= \mu(E_1) && \text{since } \lim_{k\to+\infty} f^{\frac{1}{k}}(x) = 1 \text{ for almost every } x \in E \end{aligned}$$

And $\forall x \in E_1$ and $k \in \mathbb{Z}^+$,

$$\begin{aligned} f^{\frac{1}{k}}(x) &< f(x) && \text{since } f(x) > 1 \\ \implies \lim_{k\to+\infty} \int_{E_2} f^{\frac{1}{k}} d\mu &= \int_{E_2} \lim_{k\to+\infty} f^{\frac{1}{k}} d\mu && \text{by the Dominated Convergence theorem} \\ &= \mu(E_2) && \text{since } \lim_{k\to+\infty} f^{\frac{1}{k}}(x) = 1 \text{ for almost every } x \in E \end{aligned}$$

Thus,

$$\begin{aligned} \int_E f^{\frac{1}{k}} d\mu &= \int_{E_1} f^{\frac{1}{k}} d\mu + \int_{E_2} f^{\frac{1}{k}} d\mu \\ \implies \lim_{k\to+\infty} \int_E f^{\frac{1}{k}} d\mu &= \left(\lim_{k\to+\infty} \int_{E_1} f^{\frac{1}{k}} d\mu\right) + \left(\lim_{k\to+\infty} \int_{E_2} f^{\frac{1}{k}} d\mu\right) \\ &= \mu(E_1) + \mu(E_2) \\ &= \mu(E) \end{aligned}$$