Rutgers University: Complex Variables and Advanced Calculus Written Qualifying Exam

August 2016: Problem 1 Solution

Exercise. Use a contour integral to evaluate

$$\int_0^{2\pi} \frac{d\theta}{(2+\cos(\theta))^2}$$

Solution.

$$z = e^{i\theta} \qquad dz = ie^{i\theta}d\theta \implies d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz} \qquad C : |z| = 1 \qquad e^{i\theta} \qquad 0 \le \theta \le 2\pi$$

$$\cos\theta = \frac{z - z^{-1}}{2i}$$

$$\begin{aligned}
I = \int_0^{2\pi} \frac{d\theta}{(2+\cos(\theta))^2} &= \int_{|z|=1} \frac{1}{\left(2 + \frac{z+z^{-1}}{2}\right)^2}\left(\frac{1}{iz}\right) dz \\
&= \frac{4}{i}\int_{|z|=1} \frac{1}{z(4+z+z^{-1})^2} dz \\
&= \frac{4}{i}\int_{|z|=1} \frac{z}{(4z+z^2+1)^2} dz \\
z &= \frac{-4 \pm \sqrt{16-4}}{2} \\
&= -2 \pm \sqrt{3}
\end{aligned}$$

$f(z) = \frac{z}{(4z+z^2+z^{-1})^2}$ has poles of order 2 at $z = -2 \pm \sqrt{3}$
But $-2-\sqrt{3}$ is not in the unit circle but $-2+\sqrt{3}$ is, and $z = -2-\sqrt{3}$ is a pole of order 2

$$\begin{aligned}
I &= \frac{4}{i}\left(2\pi i Res\left(f, -2+\sqrt{3}\right)\right) \\
&= 8\pi \lim_{z \to -2+\sqrt{3}} \frac{d}{dz}\left(\frac{z}{(z+2+\sqrt{3})^2}\right) \\
&= 8\pi \lim_{z \to -2+\sqrt{3}} \frac{(z+2+\sqrt{3})^2 - z(2)(z+2+\sqrt{3})}{(z+2+\sqrt{3})^4} \\
&= 8\pi \frac{(-2+\sqrt{3}+2+\sqrt{3})^2 - (-2+\sqrt{3})(2)(-2+\sqrt{3}+2+\sqrt{3})}{(-2+\sqrt{3}+2+\sqrt{3})^4} \\
&= 8\pi \frac{(2\sqrt{3})^2 + (4-2\sqrt{3})(2\sqrt{3})}{(2\sqrt{3})^4} \\
&= 8\pi \frac{12 + 8\sqrt{3} - 12}{16 \cdot 9} \\
&= \boxed{\pi \frac{4\sqrt{3}}{9}}
\end{aligned}$$