Rutgers University: Algebra Written Qualifying Exam

August 2016: Problem 2 Solution

Exercise. Let T be a square matrix over $\mathbb{C}$.

(a) Show that if T is invertible and T^k is diagonalizable for some positive integer k, then T is diagonalizable.

Solution.

T is diagonalizable $\iff$ $\exists p$ with simple roots such that $p(x) = 0$.
T^k is diagonalizable $\implies$ $\exists$a monic polynomial f s.t.

$$f(T^k) = (T^k - \lambda_1 I)(T^k -_2 I)\dots(T^k - \lambda_n I) = 0$$

and each λ_i is distinct.

Let $g(x) = f(x^k) = (x^k - \lambda_1 I)(x^k -_2 I)\dots(x^k - \lambda_n I)$.
Then $g(T) = f(T^k) = 0$.
Moreover, since T is invertible, T^k is invertible and $\lambda_i \neq 0$ for all i
$\implies$ the roots of $g(x)$ are the k^{th} roots of λ_i
$\implies$ the roots of $g(x)$ are simple
$\implies$ T is diagonalizable.

(b) Show that the invertibility hypothesis cannot be omitted in (a).

Solution.

If T is not invertible then $\lambda_i = 0$ for some i

$$\begin{aligned} g(x) &= f(x^k) \\ &= x^k(x^k - \lambda_2 I)\dots(x^k - \lambda_n I) = 0 \\ &\implies 0 \text{ is a repeated root.} \end{aligned}$$

Counterexample:

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$T \neq 0$ and has eigenvalues 0 and 0, and is not diagonalizable.

$$T^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ is a diagonal matrix.}$$