Rutgers University: Algebra Written Qualifying Exam

August 2016: Problem 1 Solution

Exercise. Let G be an abelian group and for each positive integer n, define

$$G[n] = \{g \in G | ng = 0\}.$$

(a) Show that if m and n are positive integers and m divides n, then $G[m] \subseteq G[n]$, and $G[n]/G[m]$ is isomorphic to a subgroup of $G[n/m]$.

Solution.

Since $m \mid n$, $n = dm$ for some $d \in \mathbb{N}$.
If $g \in G[m]$, then $mg = 0$

$$\implies ng = (dm)g = d(mg) = d \cdot 0 = 0$$

$$\implies g \in G[n]$$

Thus $G[m] \subseteq G[n]$.

Find an isomorphism between $G[n]/G[m]$ and some subgroup of $G[n/m]$.
First Isomorphism Theorem: If $\phi : G \to H$ is a homomorphism, then

$$G/\ker(\phi) \cong \phi(G)$$

Let $\phi : G[n] \to G$ be defined by $\phi(g) = mg$.

$$\begin{aligned} \phi(g+h) &= m(g+h) \\ &= mg + mh \\ &= \phi(g) + \phi(h) \\ \implies &\ \phi \text{ is a homomorphism} \end{aligned}$$

$\ker(\phi)$: $\quad \phi(g) = mg = 0 \iff g \in G[n] \cap G[m] = G[m]$

$Im(\phi)$: $\quad d\phi(g) = dmg = ng = 0$ for all $g \in G[n] \implies \phi(G)$ is a subgroup of $G[d]$

By the first isomoprhism theorem,

$$G[n]/G[m] \cong Im(\phi), \text{ a subgroup of } G[n/m]$$

(b) Give an example in which m divides n but $G[n]/G[m] \not\cong G[n/m]$. Prove your assertion.

Solution.

Let $G = \mathbb{Z}_{12}$.

$$G[3] = \{0, 4, 8\} \quad \text{and} \quad G[9] = \{0, 4, 8\}$$

$$G[9]/G[3] = \{\{0, 4, 8\} = G[3] + 0 = G[3] + 4 = G[3] + 8\}$$

$$\implies |G[9]/G[3]| = 1$$

But $|G[9/3]| = |G[3]| = 3$

Thus, $G[9]/G[3] \not\cong G[9/3]$