Rutgers University: Algebra Written Qualifying Exam

August 2015: Problem 3 Solution

Exercise. Let G be the group $\mathbb{Q}/\mathbb{Z}$, where $\mathbb{Q}$ and $\mathbb{Z}$ are viewed as groups under addition. Prove the following.

(a) Ever element of G has finite order.

Solution.

$$G = \mathbb{Q}/\mathbb{Z} = \{\mathbb{Z} + \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$$

Let $\mathbb{Z} + \frac{m}{n} \in G$ be arbitrary.

$$\underbrace{\left(\mathbb{Z} + \frac{m}{n}\right) + \left(\mathbb{Z} + \frac{m}{n}\right) + \cdots + \left(\mathbb{Z} + \frac{m}{n}\right)}_{n \text{ times}} = \mathbb{Z} + m = \mathbb{Z} \text{ since } m \in \mathbb{Z}.$$

Thus, $o\left(\mathbb{Z} + \frac{m}{n}\right) \mid n$ and is finite.
So, every element of G has finite order.

(b) Ever finitely generated subgroup of G is cyclic.

Solution.

Let H be a finitely generated subgroup of G with generators

$$\mathbb{Z} + \frac{m_1}{n_1}, \ldots, \mathbb{Z} + \frac{m_k}{n_k}$$

Then $H \subseteq \left\langle \mathbb{Z} + \frac{1}{n_1 n_1 \ldots n_k} \right\rangle$.
Since H is a subgroup of a cyclic group, H is cyclic.