Rutgers University: Complex Variables and Advanced Calculus Written Qualifying Exam

August 2014: Problem 4 Solution

Exercise. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^4} z^{2^n},$$

which has convergence radius 1. (Thus $f(z)$ is a well-defined holomorphic function on the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.) Prove that $f(z)$ does not admit a holomorphic extension to a neighborhood of 1 in $\mathbb{C}$, that is, there does not exist a neighborhood U of 1 in $\mathbb{C}$ and a holomorphic function g defined on U such that $f|_{U\cap\Delta} = g|_{U\cap\Delta}$.

Solution.

Take the derivative!

Suppose for contradiction $\exists$such a U and $g(z)$.

If $f|_{U\cap\Delta} = g|_{U\cap\Delta}$ then $f'|_{U\cap\Delta} = g'|_{U\cap\Delta}$.

And if g is holomorphic on a neighborhood of 1, $\lim\limits_{z\to1} g'(z)$ exists and

$$\lim_{z\to1} g'(z) = \lim_{\substack{x\to1^- \\ \in\mathbb{R}}} f'(x)$$

$$f'(z) = \sum_{n=1}^{\infty} \frac{2^n}{n^4} z^{2^n-1}$$

$$\implies \quad \lim_{x\to1^-} f'(x) = \lim_{x\to1^-} \sum_{n=1}^{\infty} \frac{2^n}{n^4} x^{2^n-1}$$

Note: we can think of this as $\lim\limits_{k\to\infty} \sum\limits_{n=1}^{\infty} \frac{2^n}{n^4} (x_k)^{2^n-1}$ since $(x_k) \to 1$ from below

Also, for all n, k $\quad \frac{2^n}{n^4} x_k^{2^n-1} < \frac{2^n}{n^4} x_{k+1}^{2^n-1}$ since $x_k < x_{k+1}$

By the **monotone convergence theorem,**

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} \frac{2^n}{n^4} (x_k)^{2^n-1} = \sum_{n=1}^{\infty} \lim_{k\to\infty} \frac{2^n}{n^4} (x_k)^{2^n-1}$$

$$= \sum_{n=1}^{\infty} \frac{2^n}{n^4}, \qquad \text{which diverges}$$

$$\implies \quad \lim_{x\to1^-} \sum_{n=1}^{\infty} \frac{2^n}{n^4} x^{2^n-1} \text{ diverges,} \qquad \text{a contradiction!}$$

Note: we could also have used **Fatou's Lemma:**

$$\lim_{x\to1^-} \inf \sum_{n\geq1} \frac{2^n}{n^4} x^{2^n-1} = \sum_{n\geq1} \lim_{x\to1^-} \inf \frac{2^n}{n^4} x^{2^n-1}$$

*These theorems are usually applied to integrals not summations. Think about when/why they work for summations.