Rutgers University: Algebra Written Qualifying Exam
August 2014: Problem 2 Solution

Exercise. Let $G = G_1 \times G_2$ where $G_1 \cong G_2 \cong S_4$, the symmetric group on four letters. Suppose that H is any subgroup of G such that $H \cong S_4$. Show that either $H \cap G_1 = 1$ or $H \cap G_2 = 1$

Solution.

Let $H_1 = H \cap G_1$ and $H_2 = H \cap G_2$

Since $G = G_1 \times G_2$, $G_1 \triangleleft G$ and $G_2 \triangleleft G$. really think $G_1 = G_1 \times \{1\}$ and $G_2 = \{1\} \times G_2$

For any normal subgroup K of G, the subgroup $K \cap H \leq H$ is normal in H

$\implies H_1 = H \cap G_1 \triangleleft H$ and $H_2 = H \cap G_2 \triangleleft H$

Also, $G_1 \cap G_2 = 1$ so $H_1 \cap H_2 = 1$

$\implies H_1 \times H_2 \cong H_1 H_2 \leq H$

And for any $x \in H$,

$$\begin{aligned} xH_1H_2x^{-1} &= (xH_1x^{-1})(xH_2x^{-1}) \\ &= H_1H_2 \qquad \text{since } H_1, H_2 \triangleleft H \end{aligned}$$

Thus, H_1H_2 is a normal subgroup of H.

Since $H_1, H_2, H_1H_2 \triangleleft H \cong S_4$, look at the normal subgroups of S_4.

$|1| = 1,$ $|S_4| = 24,$ $|A_4| = 12$ $|V_4| = 4$

Since $|H_1 \cap H_2| = 1$,

$$|H_1H_2| = |H_1||H_2|$$

By looking at possible orders of H_1, H_2, and H_1H_2, it is obvious that either

$H_1 = 1$ and $H_2 = S_4$

OR

$H_1 = S_4$ and $H_2 = 1$