Rutgers University: Algebra Written Qualifying Exam
August 2014: Problem 1 Solution

Exercise. Let V be a 5-dimensional vector space over $\mathbb{C}$ and let $T : V \to V$ be a linear transformation. Assume that there is $v \in V$ such that $\{v, Tv, T^2v, T^3v, T^4v\}$ spans V. Assume that the set of eigenvalues of T is precisely equal to $\{1, 2\}$. On the basis of this information, how many possible Jordan canonical forms are there for T, and what are they? Justify your answer.

Solution.

Since eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$, the Jordan canonical form of T has diagonal entries 1 and 2.

$$\begin{aligned} p(x) &= |A - xI| && \text{where } A \text{ is the matrix rep. of } T \text{ and } k_1 + k_2 = 5 \\ &= (x-1)^{k_1}(x-2)^{k_2} \end{aligned}$$

possible char. polynomials:

$$(x-1)(x-2)^4$$
$$(x-1)^2(x-2)^3$$
$$(x-1)^3(x-2)^2$$
$$(x-1)^4(x-2)$$

Since $\{\vec{v}, T\vec{v}, T^2\vec{v}, T^3\vec{v}, T^4\vec{v}\}$ spans V, for all $a_i \in \mathbb{C}$

$$a + T^4\vec{v} + a_3T^3\vec{v} + a_2T^2\vec{v} + a_1T\vec{v} + a_0\vec{v} = \vec{0} \iff a_0 = a_1 = a_2 = a_3 = a_4 = 0$$

If $q_A(x)$ has deg< 5 then

$$\begin{aligned} q_A(x) = b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0 \quad &\implies q_A(A) = b_4A^4 + b_3A^3 + b_2A^2 + b_1A + b_0 = \vec{0} \\ &\implies b_4A^4\vec{v} + b_3A^3\vec{v} + b_2A^2\vec{v} + b_1A\vec{v} + b_0\vec{v} = \vec{0} \\ &\implies b_i = 0 \text{ for all } i, \text{ a contradiction} \end{aligned}$$

So $\deg(q_A(x)) = 5$ and $q_A(x) = p_A(x)$ since $q_A(x) \mid p_A(x)$ and they have the same degree and are monic

If $q_A(x) = (x-1)^{d_1}(x-2)^{d_2}$, then the largest Jordan block for eigenvalue $\lambda = 1$ has size d_1,
and the largest Jordan block for eigenvalue $\lambda = 2$ has size d_2

$\implies$ since $q_A(x) = p_A(x) = (x-1)^{k_1}(x-2)^{k_2}$,
there is only one Jordan block for $\lambda = 1$ and it has size k_1
and there is only one Jordan block for $\lambda = 2$ and it has size k_2

There are 4 possible Jordan canonical forms for T

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$